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SPACES ADMITTING COMPLETE ABSOLUTE PARALLELISM*

BY L. P. EISENHART

At the Colloquium of this Society held at Ithaca in September, 1925, I set forth, under the title *The New Differential Geometry*, certain developments which had taken place during the preceding ten years, growing out of the concept of infinitesimal parallelism for Riemannian spaces proposed by Levi-Civita. When these lectures were published in book form [1]† in 1927, the book included also material which had been developed in the interim. Since that time there have been many further developments. Instead of trying to make a full survey of these, I have decided to limit my paper to the theory of linearly connected manifolds admitting a complete absolute parallelism.

Levi-Civita [2] introduced the concept of parallelism in a Riemannian space as a means of giving an invariantive interpretation to the curvature of the space. Since a Riemannian space of n dimensions, V_n, may be thought of as a sub-space of a euclidean space of suitable dimensions, Levi-Civita applied the concept of parallelism of the euclidean space to vectors tangential to the sub-space. In fact, vectors a and a' at two nearby points P and P' were defined to be parallel, if the angles between a and a tangent vector f at P and a' and f are equal from the euclidean point of view for every tangential vector f. Analytically this leads to the result that, if in terms of general coordinates x^i in V_n the coordinates of P and P' are x^i and x^i+dx^i, then ξ^i and $\xi^i+d\xi^i$ are the components of parallel directions at P and P', provided

$$(1) \qquad d\xi^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \xi^j dx^k = 0, \qquad (i, j, k = 1, \cdots, n),$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the Christoffel symbols of the second kind

* An address delivered at Atlantic City, December 28, 1932, as the retiring presidential address, before the American Mathematical Society.

† Such references are to the items in the bibliography at the end of this article.

formed with respect to the fundamental tensor g_{ij} of V_n.* Although the metrical properties of the enveloping euclidean space were used in the geometrical definition of parallelism, the analytical result involves only the quantities of V_n, that is, it is intrinsic, a result which Levi-Civita considered necessary for an appropriate definition.

Let P and Q be two nearby points of a geodesic C. Through them geodesics are drawn in parallel directions, thus making the same angle ψ with C, and on the geodesics equal lengths are laid off from P and Q with end points P' and Q', so that we have a parallelogrammoide $PQP'Q'$. If $P'Q'$ and PQ denote the lengths of these respective sides of $PQP'Q'$, then as Levi-Civita has shown [2, p. 191] the Riemannian curvature at P for the orientation determined by PQ and PP' is equal to the ratio of $(P'Q')^2-(PQ)^2$ and the square of the area of $PQP'Q'$.

Recognizing that parallelism is an affine property and thus is not limited to spaces with an assigned metric, Weyl [3] generalized equations (1) in the form

$$d\xi^i + \Gamma^i_{jk}\xi^j dx^k = 0, \tag{2}$$

where the Γ's are functions of the x's. Consider three nearby points P, P_1, and P_2 of coordinates x^i, $x^i+d_1x^i$, and $x^i+d_2x^i$. The coordinates of the end point of the vector at P_2 parallel to d_1x^i at P are

$$x^i + d_2x^i + d_1x^i + d_2d_1x^i + \Gamma^i_{jk}\, d_1x^j d_2x^k.$$

Interchanging the subscripts 1 and 2, we have the coordinates of the end point of the vector at P_1 parallel to d_2x^i at P. These points coincide when and only when

$$\Gamma^i_{jk} = \Gamma^i_{kj}, \tag{3}$$

as in the case of the Christoffel symbols. Weyl imposed this condition in his definition of infinitesimal parallelism, and of an affinely connected manifold. The quantities Γ^i_{jk} are called the *coefficients of the affine connection.*

When the coordinates x^i of the space undergo a non-singular

* Throughout this paper we use the convention that a term containing a repeated index indicates the sum of such terms as the index takes on all possible values.

transformation into coordinates x'^i, and we denote by Γ'^{i}_{jk} the coefficients of the connection in the x''s, we have

$$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + \Gamma^{i}_{jk} \frac{\partial x^j}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} = \Gamma'^{\gamma}_{\alpha\beta} \frac{\partial x^i}{\partial x'^\gamma}. \tag{4}$$

From these equations it follows that the coordinates x'^i can be chosen in many ways so that at a given point $\Gamma'^{\gamma}_{\alpha\beta}=0$, and consequently it follows from (2) that parallel vectors at P and nearby points have the same components. Consequently, in the neighborhood of a point the situation is the same as that which obtains throughout the whole of euclidean space in terms of cartesian coordinates. Moreover, this property is true only in case equations (3) are satisfied, and thus it may be taken as a characteristic property of Weyl's affine connection.

Veblen and I, [4] and [1], developed this theory by taking as basis the paths, that is, the integral curves, of the system of equations

$$\frac{d^2 x^i}{ds^2} + \Gamma^{i}_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

These curves are the straightest lines for the definition (2) of parallelism, in that the tangents to any path are parallel with respect to the path. From this basis one develops naturally a projective geometry as well as an affine one.

A necessary and sufficient condition that there exists a coordinate system for which $\Gamma'^{\gamma}_{\alpha\beta}=0$ at every point, in which case we have a euclidean space and the coordinates are cartesian, is that

$$B^{h}_{ijk} = 0, \tag{5}$$

where

$$B^{h}_{ijk} = \frac{\partial \Gamma^{h}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{h}_{ij}}{\partial x^k} + \Gamma^{l}_{ik} \Gamma^{h}_{lj} - \Gamma^{l}_{ij} \Gamma^{h}_{lk}. \tag{6}$$

If in the equations

$$\frac{d\xi^i}{dt} + \Gamma^{i}_{jk} \xi^j \frac{dx^k}{dt} = 0 \tag{7}$$

we put

$$x^i = \phi^i(t), \tag{8}$$

a solution of the system (7) is determined by the initial values of the ξ's. Thus at each point of the curve (8) we have a vector and all of these vectors are said to be parallel. Consequently, we have distant parallelism of vectors, but in general, as follows from (7), the vector at a point Q parallel to a vector at a point P depends upon the curve joining P and Q. Thus the parallelism is not absolute as in the case of euclidean space. Levi-Civita introduced this idea, using (7) in which the Γ's are the Christoffel symbols, and Weyl generalized it to affine connections.

If parallelism is to be independent of the curve, ξ^i must satisfy the equations

$$\frac{\partial \xi^i}{\partial x^k} + \Gamma^i_{jk}\xi^j = 0, \quad (i, j, k = 1, \cdots, n). \tag{9}$$

Each set of solutions of these equations defines a field of absolutely parallel vectors. In order that there may be n independent fields, it is necessary that the conditions of integrability of the above equations, that is, $\xi^i B^k_{ijk} = 0$, be satisfied identically. Consequently, equations (5) must hold and the space be euclidean.

We have remarked that Weyl imposed the condition that the functions Γ^i_{jk} be symmetric in j and k in order to insure that in the neighborhood of a point parallelism be euclidean. If this restriction is not made, we say that the affine connection is *asymmetric* and take as the coefficients L^i_{jk}, denoting by Γ^i_{jk} and Ω^i_{jk} the symmetric and asymmetric parts, so that

$$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}. \tag{10}$$

In place of (4) we have

$$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} + L^i_{jk}\frac{\partial x^j}{\partial x'^\alpha}\frac{\partial x^k}{\partial x'^\beta} = L'^\gamma_{\alpha\beta}\frac{\partial x^i}{\partial x'^\gamma}, \tag{11}$$

from which follow (4) and

$$\Omega^i_{jk}\frac{\partial x^j}{\partial x'^\alpha}\frac{\partial x^k}{\partial x'^\beta}\frac{\partial x'^\gamma}{\partial x^i} = \Omega'^\gamma_{\alpha\beta}, \tag{12}$$

so that the Ω's are components of a tensor. Any set of functions L^i_{jk} and $L'^\gamma_{\alpha\beta}$ satisfying (11) are said to determine an asymmetric

affine connection. If in (9) we replace Γ^i_{jk} by L^i_{jk}, we have the equations of condition that a field of vectors be absolutely parallel and their condition of integrability is $\xi^i L^h_{ijk}=0$, where L^h_{ijk} is the same expression in the L's as B^h_{ijk} (6) is in the Γ's. Consequently a necessary and sufficient condition that there be n independent fields of parallel vectors is that

$$L^h_{ijk} = 0;$$

in this case we say that the space admits *complete absolute parallelism*, and that it is of *zero curvature*. Since Ω^i_{jk} are components of a tensor, if they are zero in one coordinate system at a point they are zero in every system. Hence when, and only when, the connection is symmetric is it possible for all the coefficients of the connection to be zero at a point. Consequently spaces with asymmetric connections of zero curvature are not euclidean, but pseudo-euclidean in that they admit complete absolute parallelism.

If in the case of a space of zero curvature we denote by λ_α the components of n independent fields of parallel vectors, i indicating the component and α the vector, we have

$$\frac{\partial \lambda^i_\alpha}{\partial x^k} + \lambda^j_\alpha L^i_{jk} = 0. \tag{13}$$

Since the matrix $\|\lambda^i_\alpha\|$ is of rank n, quantities λ^α_i are defined uniquely by

$$\lambda^i_\alpha \lambda^\alpha_j = \delta^i_j, \quad \lambda^\beta_\alpha \lambda^\beta_i = \delta^\beta_\alpha, \tag{14}$$

where a δ is equal to one or zero, according as its indices are the same or different. By means of these functions we have from (13)

$$L^i_{jk} = -\lambda^\alpha_j \frac{\partial \lambda^i_\alpha}{\partial x^k} = \lambda^i_\alpha \frac{\partial \lambda^\alpha_j}{\partial x^k}. \tag{15}$$

Conversely, if we have n independent vectors λ^i_α and define L^i_{jk} by (15) with the aid of λ^α_i, the quantities so determined in any two coordinate systems satisfy (11), and thus determine an affine connection with respect to which the vectors λ^i_α are absolutely parallel. In 1922–1923 Weitzenböck [5, p. 319] showed that the functions (15) satisfy (11) and used them as a basis for

covariant differentiation, but made no reference to the parallelism which they define. However, in 1923 Vitalli [6] used them to define parallelism, and pointed out that each field of vectors is absolutely parallel.

In 1917 Hessenberg [7] developed a differential geometry based upon an ennuple of independent vectors and developed a type of differentiation based thereon. Two years later König [8] analyzed these concepts in the light of Weyl's contribution and called attention to the fact that the affine connection defined thereby is not symmetric. So far as I know this is the first reference to the use of an asymmetric connection, although evidently it was considered by Weyl and not used for the reasons previously mentioned. In 1922–1923 Cartan investigated the geometry of a space with an affine connection based on an ennuple of vectors and developed the concept of equipollance involving the idea of parallelism. He showed that the vanishing of the curvature tensor is the condition for complete absolute parallelism [9, p. 368] and thus he seems to have been the first writer to deal explicitly with this concept. Schouten [10] made an extensive study of types of linear connection.

If we put

$$g_{ij} = \lambda_i^\alpha \lambda_j^\alpha, \quad g^{ij} = \lambda_\alpha^i \lambda_\alpha^j, \tag{16}$$

we have

$$g_{ij} g^{jk} = \delta_j^k .$$

The tensor g_{ij} thus defined may be taken as the basis for a Riemannian metric in the affinely connected manifold, in which case the vectors λ_α^i form an orthogonal ennuple. The Christoffel symbols formed with respect to g_{ij} are in the following relation to the coefficients of the affine connection:

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \Gamma_{jk}^i + g^{ih}(g_{jl}\Omega_{hk}^l + g_{kl}\Omega_{hj}^l) . \tag{17}$$

Vitalli [6] proposed this introduction of a metric.

What I have presented thus far was a matter of record, when in 1928 Einstein [11] proposed a unified theory of gravitation and electricity based upon the concept of a Riemannian space admitting distant parallelism. He was unaware of the existence of the requisite mathematical knowledge and developed it

anew. He said "The new unitary field theory is based on the following mathematical discovery: There are continua with a Riemannian metric and distant parallelism which nevertheless are not euclidean." Later he gave up hope of founding a satisfactory theory on such a basis and recently turned to what is essentially a generalized projective geometry. The elements of such a geometry were presented in my colloquium lectures, and I might have chosen this subject for my address. But this theory and its application to relativity are presented by Veblen in his forth-coming book, *Projektive Relativitätstheorie*, one of Springer's "Ergebnisse der Mathematik."

Bianchi, [**12,** p. 517], showed that a simply transitive group is the group of motions of determinate Riemannian manifolds. In 1925 I [**13**] gave a new proof of this result, making use of the linear connection whose coefficients are defined by

$$L^i_{jk} = -\xi^\alpha_k \frac{\partial \xi^i_\alpha}{\partial x^j} = \xi^i_\alpha \frac{\partial \xi^\alpha_k}{\partial x^j}, \tag{18}$$

where ξ^i_α are the vectors of the transitive group and ξ^α_i are given by equations of the form (14), and showed that the connection of coefficients L^i_{jk} has zero curvature. Consequently the equations

$$\frac{\partial g_{ij}}{\partial x^k} = g_{il} L^l_{jk} + g_{jl} L^l_{ik} \tag{19}$$

are completely integrable, and it can be shown that any solution is the fundamental tensor of a Riemannian space admitting the group as a group of motions. Since the connection is of zero curvature, there are n independent fields of absolutely parallel vectors, say λ^i_α, in terms of which L^i_{jk} are expressed in the form (15). The quantities g_{ij} given by (16) satisfy (19), and each solution of (19) is expressible in terms of suitable linear combinations of the λ's with constant coefficients. We remark furthermore that the λ's are the vectors of the simply transitive group reciprocal to the given group. They may be used as in (18) to determine a second affine connection of zero curvature, whose coefficients $\bar{L}^i_{jk}$ satisfy the condition $\bar{L}^i_{jk} = \bar{L}^i_{kj}$.

When an affine connection has zero curvature, an ennuple of absolutely parallel vectors are the vectors of a simply transitive

group, when and only when the first covariant derivatives of Ω^i_{jk} with respect to the L's are zero. In this case Γ^i_{jk} and Ω^i_{jk} satisfy the conditions

$$B^h_{ijk} = \Omega^l_{jk}\Omega^h_{il}.$$

In accordance with the theory of Lie, if

$$x'^i = f^i(x^1, \cdots, x^n; a^1, \cdots, a^r)$$

are the equations of an r-parameter continuous group, then

$$\frac{\partial x'^i}{\partial a^t} = \xi^i_\alpha(x')A^\alpha_t(a), \qquad \begin{pmatrix} i = 1, \cdots, n; \\ \alpha, t = 1, \cdots, r \end{pmatrix},$$

where ξ^i_α are the vectors of the group, and under a non-singular transformation of the a's the A's are related as covariant vectors. They may be interpreted as vectors in the group-space S of coordinates a^t. By means of equations similar to (14) we obtain an ennuple of contravariant vectors A^t_α, and as is well known they are the vectors of a simply transitive group. Consequently they, and the vectors of the reciprocal group, determine two affine connections of zero curvature for the group space. This theory has been studied extensively by Cartan [14, 15] and Schouten [14, 16] as setting forth the geometry of the two-parameter groups of a given group. In particular the trajectories of a one-parameter sub-group of either parameter group are parallel paths of S.

Einstein and Mayer [17] in developing the unified theory on the basis of absolute parallelism considered the rotation group of motions of the manifold in order to determine a spherically-symmetric space-time. Robertson [18] considered the general question of motions of a linearly connected manifold of zero curvature. His definition is equivalent to saying that if equations (11), in which L'^i_{jk} are the same functions of the x''s as the corresponding L^i_{jk} are of the x's, admit a solution involving one or more parameters, these solutions define a motion. When a metric is assigned, as in (16), the equations of motion imply the Killing equations. In this case a group of maximum order, namely $n(n+1)/2$, is realized only in case $\Omega^i_{jk}=0$ and the space is euclidean, except when $n=1$ or 3. In the latter case the space is of constant curvature. If the curvature is positive and the

metric is positive definite, the two parallelisms are those discovered long ago by Clifford. Cartan [**19**, p. 704] had shown previously that the two types of parallelism of Clifford could be identified with parallelism in a linearly connected manifold of zero curvature.

The differential invariants of a linearly connected manifold of zero curvature have been studied by Griss [**20**], Weitzenböck [**21**], and Bortolotti [**22**], but we shall not set forth their results.

In presenting this review of the development of the concept of complete absolute parallelism I may quite unintentionally have omitted references to writers which should be included. Furthermore I have not attempted to give a complete bibliography of the subject; the reader will find further references in the papers listed, and in particular in [**21**], [**23**], and [**24**].

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PRINCETON UNIVERSITY

THE PROBLEM OF PLATEAU†

BY JESSE DOUGLAS

1. *Formulation.* The problem of Plateau is to prove the existence of a minimal surface bounded by a given contour. The first and only complete solution of this problem was found by the present author [1–11]‡; in this solution the contour is an arbitrary Jordan curve in n-dimensional euclidean space. Besides this generality of result, the chief contribution of the work seems to lie in the introduction of a certain new functional $A(g)$, which furnishes the key to the Plateau problem and makes clear its relationship to other fundamental problems of analysis.

For definition of minimal surface, we take the formulas of Weierstrass,

$$x_i = \Re F_i(w), \quad \sum_{i=1}^{n} F_i'^2(w) = 0, \tag{1}$$

though Weierstrass himself considered only the case $n=3$. Here w is a complex variable, $w=u+iv$, of which the F_i are monogenic functions. When $n=2$, these formulas become

$$x_1 = \Re F_1(w), \quad x_2 = \Re F_2(w), \quad F_1'^2(w) + F_2'^2(w) = 0, \tag{2}$$

or

$$F_2'(w) \pm iF_1'(w) = 0. \tag{3}$$

If

$$F_1(w) = P_1(u, v) + iQ_1(u, v), \quad F_2(w) = P_2(u, v) + iQ_2(u, v),$$

then

$$F_1'(w) = \frac{\partial P_1}{\partial u} + i\frac{\partial Q_1}{\partial u} = \frac{\partial P_1}{\partial u} - i\frac{\partial P_1}{\partial v},$$

$$F_2'(w) = \frac{\partial P_2}{\partial u} + i\frac{\partial Q_2}{\partial u} = \frac{\partial P_2}{\partial u} - i\frac{\partial P_2}{\partial v},$$

† Address delivered by invitation of the program committee at the New York meeting, October 29, 1932.

‡ Numerical references are to the bibliography at the end.

where we have used the Cauchy-Riemann equations for P_1, Q_1, and P_2, Q_2. If we use the plus sign,† (3) implies that

$$\frac{\partial P_1}{\partial u} = \frac{\partial P_2}{\partial v}, \quad \frac{\partial P_1}{\partial v} = -\frac{\partial P_2}{\partial u},$$

which are the Cauchy-Riemann equations for the functions (2), that is, these functions are conjugate harmonic. The equations (2) therefore represent a conformal map of the plane x_1+ix_2 on the w plane.

Consequently the problem of Plateau, as I have formulated and solved it, contains the conformal mapping problem of Riemann: to map conformally a given plane region on the interior of a circle (in the very general case of any region bounded by a Jordan curve). It will include also the theorem of Osgood [31] and Carathéodory [32] to the effect that the map remains one-to-one and continuous as between the circumference of the unit circle and the Jordan curve bounding the given region.

2. *Historical.* The Plateau problem is named for the Belgian natural scientist J. Plateau [27], who made numerous experiments with soap films, realizing a large variety of minimal surfaces. A soap film, under its surface tension, takes the form of least area consistent with its constraints, that is, with the condition of being bounded by a given contour; and this least area property is a characteristic of minimal surfaces.

It was in this way, indeed, that minimal surfaces first appeared in mathematics, in a pioneer memoir of Lagrange (1760) [28] on the calculus of variations for double integrals. The analytic formulation of the problem of least area is

$$\iint (1 + p^2 + q^2)^{1/2} dx dy = \text{minimum},$$

among all surfaces $z=f(x, y)$ with a given boundary Γ, and the condition found by Lagrange for the minimizing surface is

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0, \tag{*}$$

where p, q, and r, s, t denote, as usual, the first and second partial derivatives of z. Some years later (1776), Meusnier in-

† The minus sign leads to an inversely conformal transformation.

terpreted this equation geometrically as expressing zero mean curvature:

$$\frac{1}{R_1}+\frac{1}{R_2}=0.$$

With the use of complex quantities, Monge (1784) integrated the partial differential equation of minimal surfaces, applying the parametric form of representation instead of the restricted form $z=f(x, y)$, and the formulas (1) of Weierstrass (1866) are essentially a transformation of those of Monge.

The problem of Plateau was taken up by Riemann [23], Weierstrass [24], and Schwarz [25], who confined themselves in the main to the case where the given contour is a skew polygon. They made the problem depend on a monodromy-group problem associated with a linear differential equation of the second order

$$\frac{d^2\theta}{dw^2}+p\frac{d\theta}{dw}+q\theta=0,$$

whose coefficients p and q are rational functions of w; but they did not succeed in obtaining a complete solution even for the polygonal case.

S. Bernstein (1910) [21] and A. Haar (1927) [20] did important work on the problem, using the restricted representation $z=f(x, y)$ of the surface. Bernstein took for basis the elliptic partial differential equation (4), and considered the Plateau problem as a generalized Dirichlet problem with the equation (4) taking the place of Laplace's equation $r+t=0$. Haar used the direct methods of the calculus of variations introduced by Hilbert. Both writers assumed the given contour to have a convex projection on the xy-plane.

The case of a contour of general shape has been considered only in recent years. R. Garnier (1928) [19] followed the classic methods of Riemann and Weierstrass, and concluded the existence of a minimal surface bounded by any given contour of (piece-wise) bounded curvature. Besides the general difficulty of following Garnier's developments to this conclusion, it is to be remarked that his work is based on that of Weierstrass, who assumes explicitly that the contour has no knots.† It is

† Weierstrass, *Werke*, vol. 3, p. 220, last line.

therefore hard to see how Garnier frees himself of this restriction.

After the publication of the abstracts of the present author's papers in this Bulletin [1–10], T. Radó obtained by other methods results, which, as far as the Plateau problem is concerned, were not as general as those already found by the present author. Radó used the conformal mapping of polyhedral surfaces approximating to the required minimal surface, and restricted the contour first to be rectifiable [17], and then [18] to be capable of bounding a simply-connected surface of finite area.

The author, in collaboration with P. Franklin [16], has constructed an example of a step-polygon of a denumerable infinity of sides, every surface bounded by which has infinite area. Recently, I have constructed even simpler examples of Jordan space curves which bound no orientable surface of finite area†, for instance, the spiral defined in spherical coordinates by the equations $r=\cos\phi$, $\theta=\tan^5\phi$. The simplicity of these examples shows that one must be careful not to overestimate the generality of a contour that spans a finite area.

The method of solution here explained was developed by the author in the years 1926–1929. Besides the abstracts in this Bulletin, the essential features of the work were widely publicized by the author's lectures in mathematical seminars at Paris, Göttingen, and Hamburg during the year 1929. A comprehensive and didactic presentation was published in the Transactions of this Society in January, 1931, [11], followed by papers extending the theory to two contours [12] and to one-sided minimal surfaces [13].

The advantages peculiar to the present treatment may be recapitulated as follows:

(i) The contour is subject to no restrictions whatever, except that it is any Jordan curve. The number of dimensions n of the containing euclidean space is indifferent.

(ii) The conformal mapping theorem of Riemann for any Jordan region is included as the special case $n=2$.

(iii) The conformal map is derived from a topological correspondence between the boundaries, so that the theorem of Osgood and Carathéodory falls out as an immediate corollary.

† Proceedings of the National Academy of Sciences, vol. 19 (1933), Nos. 2, 4.

3. *The Weierstrass Proof of Attainment of a Minimum.* The point and specific difficulty in the problem of Plateau is to prove that a certain minimum is attained, a minimum of area. The first to give a rigorous proof of the attainment of a minimum was Weierstrass, when he showed that every continuous function of a real variable on a closed interval attains its maximum and its minimum values.

The definition of a continuous function $f(x)$ is that if $x_1, x_2, \cdots, x_n, \cdots$ is any infinite sequence of points converging to a point x^*, so that $\lim_{n\to\infty} x_n = x^*$, then $\lim_{n\to\infty} f(x_n) = f(x^*)$.

The property of continuity may be divided into two properties, lower and upper semi-continuity. A function is lower semi-continuous if when

$$\lim_{n\to\infty} x_n = x^*, \text{ and } \lim_{n\to\infty} f(x_n) = L, \tag{5}$$

then

$$f(x^*) \leqq L. \tag{6}$$

Similarly, a $\geqq$ sign in the last relation gives the definition of upper semi-continuous.

The theorem that a continuous function on a closed interval attains its extreme values may be divided into two parts: a lower semi-continuous function attains its minimum; an upper semi-continuous function attains its maximum. We need consider only the first of these, since the two are evidently equivalent by the transformation of $f(x)$ into $-f(x)$. Let us examine the simple details of its proof.

Let m be the lower bound of the function $f(x)$ in the interval (ab). The existence of m as a finite quantity is implied by the lower semi-continuity of $f(x)$,† but often, as in the present application, it is known a priori that m exists and is finite, for instance, when $f(x)$ is an essentially positive quantity. Then, by definition of lower bound, we can always construct a "minimizing sequence"

$$x_1, x_2, \cdots, x_n, \cdots, \tag{7}$$

† This results from the present proof that m, finite or $-\infty$, is attained, that is, m is one of the values of the function; and every value of $f(x)$ is supposed finite.

such that

$$f(x_1), f(x_2), \cdots, f(x_n), \cdots \to m. \tag{8}$$

It is to be noted that it is only the functional values which tend to a limit; the argument values need not do so. Now we can apply the theorem of Bolzano-Weierstrass to the effect that the infinite sequence of points (7) on the finite closed interval (ab) must contain a convergent subsequence

$$x_1', x_2', \cdots, x_n', \cdots \to x^*. \tag{9}$$

This property of any infinite set of points of the interval (ab) is expressed by saying that this interval is a *compact* set of points. Of course, the limit point x^* belongs to (ab), since (ab) is also a closed set. Together with (9), we have evidently

$$f(x_1'), f(x_2'), \cdots, f(x_n'), \cdots \to m, \tag{10}$$

since (10) is a subsequence of (8); therefore, by the definition (6) of lower semi-continuity, it follows that $f(x^*) \leqq m$. But $f(x^*) < m$ is impossible, by definition of lower bound; consequently $f(x^*) = m$, which was to be proved; the minimum of $f(x)$ is attained at x^*. Such is the simple Weierstrass proof for the attainment of a minimum. It will be observed that it requires only the following two conditions:

(i) The argument range R is *compact* and *closed*; that is, every infinite sequence of elements of R contains a subsequence converging to a limit, which is also an element of R.

(ii) The function f is *lower semi-continuous*.

A very general point of view as to functions on abstract sets of elements of any nature was developed by M. Fréchet [29] in his thesis, and the proof of attainment of a minimum was given by him as a consequence solely of the postulates (i) and (ii).

The present writer's contribution to the problem of Plateau is essentially this: *by means of the new functional $A(g)$, to have given the problem an analytic formulation to which the Weierstrass-Fréchet proof for attainment of a minimum applies immediately.* No treatment theretofore proposed does this.

That *the functional $A(g)$ also unites the Plateau and Riemann problems of minimal surfaces and conformal mapping*, is an additional mark of its appositeness and importance, since that, too, was not accomplished by any previous point of view.

4. *Comparison of Three Classic Problems.* The classic analytic setting of the problem of Plateau was as a least area problem, the argument range consisting of all surfaces $[\Sigma]$ bounded by the given contour Γ, and the functional to be minimized being $S(\Sigma)$, the area of Σ. But to this formulation the Weierstrass-Fréchet reasoning does not apply because the range $[\Sigma]$ is not compact.

It will be instructive for purposes of comparison to consider also the Dirichlet problem. This is the problem of the existence of a function $\phi(x, y)$, taking prescribed values on the boundary of a given plane region and obeying in the interior of the region Laplace's equation $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=0$. This equation expresses the vanishing of the first variation of the integral

$$D(\phi) = \int\int\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]dxdy$$

taken over the region, so that the problem is equivalent to the minimizing of $D(\phi)$ in the totality of functions of class C''† taking the prescribed boundary values.

The conformal mapping of a region R on the interior of a circle depends on the solution of a Dirichlet problem for R. Let $G(x, y; x_0, y_0)$ denote the Green's function for R. By definition, we have

$$G(x, y; x_0, y_0) = g(x, y; x_0, y_0) + \log r,$$

where g is regular harmonic in R, $r^2=(x-x_0)^2+(y-y_0)^2$, (x_0, y_0) denotes an arbitrary point of R, and G is required to reduce to zero on the boundary of R. This implies that $g(x, y; x_0, y_0) = -\log r$ on the boundary, so that the determination of the Green's function for R consists of the solution of a certain Dirichlet problem. The Green's function having been found, the conformal mapping function can be written immediately as e^{G+iH}, where H is the conjugate harmonic function to G.

Riemann believed he could assert the existence of G, or the solution of the associated Dirichlet problem, by the following argument: $D(\phi)$ is always a positive number; *let ϕ^* be the function with the assigned boundary values for which $D(\phi)$ is least*; then ϕ^* obeys Laplace's equation. Weierstrass pointed out the

† Having continuous first and second partial derivatives.

gap in this reasoning, which consists in the assumption involved in the italicized statement that there exists a function ϕ^* for which the minimum of $D(\phi)$ is attained. The cogency of the Weierstrass objection rests on the fact that the range of the argument $[\phi]$, consisting of all functions of class C'' in R and with prescribed boundary values, is not compact. We can start by forming a minimizing sequence $\phi_1, \phi_2, \cdots, \phi_n, \cdots$, with

$$D(\phi_1), D(\phi_2), \cdots, D(\phi_n), \cdots \to \min D(\phi),$$

but nothing guarantees that this minimizing sequence contains a subsequence converging to a function of class C'' (or even converging at all). If this difficulty could be passed, the lower semi-continuity of $D(\phi)$ could then be used to prove the attainment of the minimum.

The typical and essential feature of the author's work on the Plateau problem is the introduction of a *lower semi-continuous* functional $A(g)$ whose argument range is *compact.* This functional is

$$A(g) = \frac{1}{4\pi}\int_C\int_C \frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2}{4\sin^2\dfrac{\theta-\phi}{2}}\,d\theta d\phi, \tag{11}$$

whose argument g is an arbitrary parametric representation $x_i = g_i(\theta)$ of the given contour Γ as topological image of the unit circumference C.

The positive number thus associated with any topological correspondence between Γ and C may be interpreted as proportional to the mean square of the ratio of a chord of Γ to the corresponding chord of C. The topological transformation g^* of C into Γ which minimizes $A(g)$ and solves the Plateau problem may accordingly be regarded as the one which, on the average, brings the points of C closest together [33].

The connection of the functional $A(g)$ with the Plateau problem is that if

$$x_i = \Re F_i(w) \tag{12}$$

denote the harmonic functions in the interior of the unit circle determined by the boundary values $g_i(\theta)$, which are given by

Poisson's integral

$$F_i(w) = \frac{1}{2\pi}\int_C g_i(\theta)\frac{e^{i\theta}+w}{e^{i\theta}-w}\,d\theta, \tag{13}$$

then the vanishing of the first variation of $A(g)$ expresses that $\sum_{i=1}^{n} F_i'^2(w)=0$, which is the condition for a minimal surface.

The integral which expresses $A(g)$ is improper, on account of the singularity when $\theta=\phi$; but since the integrand is positive, $A(g)$ always has a definite value, finite and positive or $+\infty$. In the main body of the work we assume that $A(g)$ is not identically equal to $+\infty$; or, with the definition $m(\Gamma)=\min A(g)$, that $m(\Gamma)$ is finite. A sufficient condition for this is that Γ be rectifiable. A more concrete interpretation of $m(\Gamma)$ is as the minimum area of all simply-connected surfaces bounded by Γ.† Afterwards, by an easy limit process, the existence proof is extended to the case $m(\Gamma)=+\infty$ of an arbitrary Jordan contour.

The following is a summary of the relationships between the three problems we have been comparing.

A. *Dirichlet.* (a) The argument range $[\phi]$ consists of all functions of class C'' in a given region R with assigned values on the boundary Γ of R.

(b) The functional is

$$D(\phi) = \int\int_R \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] dx dy.$$

(c) $D(\phi)$ is lower semi-continuous (positive integrand).

(d) $[\phi]$ is *not* compact.

(e) The Weierstrass-Fréchet reasoning *cannot* be applied to prove that min $D(\phi)$ is attained.

(f) $\delta D(\phi)=0$ is Laplace's equation.

B. *Least Area.* (a) The argument range $[\Sigma]$ consists of all surfaces of class C'' bounded by a given contour Γ.

(b) The functional is the area of Σ:

$$S(\Sigma) = \int\int (EG - F^2)^{1/2} du dv.$$

† With this interpretation, $m(\Gamma)$ was first defined by Lebesgue [22], p. 304.

(c) $S(\Sigma)$ is lower semi-continuous (positive integrand).

(d) $[\Sigma]$ is *not* compact.

(e) The Weierstrass-Fréchet reasoning *cannot* be applied to prove that min $S(\Sigma)$ is attained.

(f) $\delta S(\Sigma)=0$ is $1/R_1+1/R_2=0$, which expresses the vanishing of the mean curvature of Σ.

C. *Plateau-Riemann.* (a) The argument range $[g]$ consists of all parametric representations of the given contour Γ, or all topological correspondences between Γ and the unit circumference C.

(b) The functional is

$$A(g)=\frac{1}{4\pi}\int_C\int_C\frac{\sum_{i=1}^{n}[g_i(\theta)-g_i(\phi)]^2}{4\sin^2[(\theta-\phi)/2]}\,d\theta d\phi.$$

(c) $A(g)$ is lower semi-continuous (positive integrand).

(d) $[g]$ is *compact.*

(e) Therefore, by the Weierstrass-Fréchet reasoning, *the minimum $A(g)$ is attained.*

(f) $\delta A(g)=0$ is $\sum_{i=1}^{n}F_i'^2(w)=0$, which expresses that the harmonic surface $x_i=\Re F_i(w)$ determined by the boundary values $x_i=g_i(\theta)$ is minimal and in conformal representation on the interior of the unit circle. For $n=2$, this gives the Riemann mapping theorem, together with the theorem of Osgood and Carathéodory.

5. *Solution of the Problem of Plateau.* OUTLINE. We shall first outline the successive steps in the solution, and then give the details of proof. An arbitrary parametric representation of Γ can be defined by writing some fixed initial parameter t, which may be supposed to vary from 0 to 2π, as a monotonic increasing continuous function of θ, the polar angle on the unit circle: $t=\mu(\theta)$. Since all these functions μ have the same total variation 2π, their set is compact, according to a theorem of Helly.† To obtain closure of the set, we must however introduce *improper*

† An infinite set of functions is *compact* if the functions are uniformly bounded and of uniformly bounded variation. As stated in geometric form by Fréchet [29], an infinite set of curves all contained in a finite region and of uniformly bounded lengths is compact. This theorem goes back to Hilbert, in his work on the Dirichlet principle, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899), p. 184.

representations g, corresponding to functions μ whose graph contains line-segments parallel to the t or θ axis. These cause a partial arc of Γ to correspond to a point of C (first kind), or a partial arc of C to a point of Γ (second kind); finally, there is the extreme case of a representation causing all of Γ to correspond to a single point of C and all of C to a single point of Γ (degenerate representation). Graphically, this is represented by a line segment of length 2π along the θ axis joined to a line segment of length 2π parallel to the t axis, and must be included for closure of the set since evidently it can be obtained as a limit of proper monotonic graphs.

It will follow easily from the positive nature of the integrand of $A(g)$ that this functional is lower semi-continuous; hence, by the Weierstrass-Fréchet reasoning, the minimum of $A(g)$ is attained for a certain representation g^*.

The surface $x_i = \Re F_i(w)$ determined by g^* according to the formula (13) obeys the condition $\sum_{i=1}^{n} F_i'^2(w) = 0$ characteristic of a minimal surface, for this condition, as will be proved, expresses that $\delta A(g) = 0$ for $g = g^*$.

It remains to exclude the possibility that g^* be improper or degenerate. The representation g^* cannot be improper of the first kind, for it will be shown that then $A(g) = +\infty$, whereas, by assumption, $A(g)$ sometimes takes finite values. The case g^* improper of the second kind will be ruled out with the help of the condition $\sum_{i=1}^{n} F_i'^2(w) = 0$.

The possibility that g^* is degenerate will be prevented by the observation that $A(g)$ is invariant under linear fractional transformation of the unit circumference into itself:

$$\tan\frac{\theta'}{2} = \frac{a\tan(\theta/2) + b}{c\tan(\theta/2) + d},$$

involving three arbitrary constants $a:b:c:d$. Therefore no generality is lost if we fix three distinct points P_1, P_2, P_3 of C and three distinct points Q_1, Q_2, Q_3 of Γ, and consider only those topological correspondences between C and Γ which cause these points to correspond respectively. Then at most the sum of two of the arcs P_1P_2, P_2P_3, P_3P_1 can correspond to a point of Γ, and the sum of two of the arcs Q_1Q_2, Q_2Q_3, Q_3Q_1 to a point of C. The degenerate representations are thus excluded.

The minimizing representation g^* is therefore proper; consequently the minimal surface determined by it is bounded by Γ, for Poisson's integral attaches continuously to its boundary values at every point θ where these values are continuous, and a proper representation is everywhere continuous.

This is the outline of the solution of the Plateau problem for all contours with finite $m(\Gamma)$. The case $m(\Gamma) = +\infty$ will then be dealt with by means of a simple limit process.

We proceed to develop the details. First, we write an expression for $A(g)$ in terms of complex variables:

$$A(g) = \frac{1}{4\pi}\int_C \int_C \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \frac{dz\,d\zeta}{(z-\zeta)^2}, \tag{14}$$

where

$$z = e^{i\theta},\ \zeta = e^{i\phi} \tag{15}$$

vary over the unit circumference C, regarded as contained in the complex plane. The identity of (14) with (11) is immediate, by the substitution (15). Another way of writing (14) is

$$A(g) = \frac{1}{4\pi}\int_C \int_C \sum_{i=1}^{n} [g_i(z) - g_i(\zeta)]^2 \cdot (z, z+dz, \zeta, \zeta+d\zeta), \tag{16}$$

where the parenthesis denotes an anharmonic ratio. From this it is evident that, although complex quantities enter into the element of integration, its value is real and positive, since z, $z+dz$, ζ, $\zeta+d\zeta$ are concyclic points and the first pair is not separated by the second. This positive real character is, of course, also known through the identity of (14) with (11).

It follows immediately from the form (16) that $A(g)$ is invariant under any linear fractional transformation

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \tag{17}$$

where α, β, γ, δ are any complex quantities such that $\alpha\delta - \beta\gamma \neq 0$. This is the invariance property of $A(g)$ used in the preceding outline to avoid the degenerate representations. The transformation (17) also shows that any circle C in the complex plane may be used instead of the unit circle on which to represent Γ parametrically, $A(g)$ being defined by (14) or (16).

LOWER SEMI-CONTINUITY. That the functional $A(g)$ is lower semi-continuous is to be seen as follows. We have to define $A(g)$ as an improper integral by the formula $A(g)=\lim_{\epsilon\to 0} A_\epsilon(g)$, where $A_\epsilon(g)$ is defined by (14) with the variables of integration restricted by $|z-\zeta|\geqq\epsilon$. The functional $A_\epsilon(g)$ is, for every fixed $\epsilon>0$, continuous, which means that if g is a variable representation of Γ which approaches a certain fixed representation as limit, then $\lim A_\epsilon(g)=A_\epsilon(\lim g)$. This results from a theorem of Lebesgue, according to which one may pass to the limit under the sign of integration provided the integrand remains uniformly bounded; and a uniform upper bound for the absolute value of the integrand of $A_\epsilon(g)$ is d^2/ϵ^2, where d is the diameter (greatest chord) of Γ.

The approach of $A_\epsilon(g)$ to $A(g)$ as $\epsilon\to 0$ is in a monotonic increasing way, since diminishing ϵ adds positive elements to the integral. It is easy to prove the following lemma.†

A functional is lower semi-continuous if it can be represented as the limit of a continuous functional which tends to it in increasing.

Hence $A(g)$ is lower semi-continuous. This, with the compactness and closure of the set $[g]$, proves the existence of g^* for which the minimum of $A(g)$ is attained.

EXCLUSION OF IMPROPER REPRESENTATIONS OF THE FIRST KIND. To exclude the possibility that g^* is improper of the first kind, let $l>0$ be the length‡ of the chord of the arc of Γ which corresponds in the improper representation to the point P of C; then if z and ζ are on opposite sides of P and approach to it, the integrand of $A(g)$ has asymptotically the form $[l^2/(z-\zeta)^2]dzd\zeta$, of which the indefinite integral is $l^2\log(z-\zeta)$, and this becomes infinite for $z=\zeta$, giving the result $A(g)=+\infty$.

We now know that g^* is proper or, at the worst, improper in such a manner that it remains constantly at a fixed point of Γ while z describes a partial arc of C. Accordingly, g^* is, in any event, continuous. We can then form, with the boundary values $g=g^*$, Poisson's integral:

$$(18)\qquad x_i=\Re F_i(w),\qquad F_i(w)=\frac{1}{2\pi i}\int_C g_i(z)\frac{z+w}{z-w}\frac{dz}{z},$$

† [11], p. 282.

‡ In the condition $l>0$, we are using the assumption that the contour has no double points; the end-points of any partial arc of Γ are distinct.

which will define a harmonic surface bounded by Γ.

VANISHING OF THE FIRST VARIATION OF $A(g)$. The point now is that the vanishing of the first variation of $A(g)$ for $g=g^*$,

$$\delta A(g) = 0, \tag{19}$$

expresses exactly the condition

$$\sum_{i=1}^{n} F_i'^2(w) = 0, \tag{20}$$

which characterizes as a minimal surface the one defined by the formulas (18). After (20) has been established, it will be easy to get rid of the remaining possibility that g^* be improper of the second kind.

In proceeding to the proofs, we first derive from (18), by a permissible differentiation under the integral sign, the formula

$$F_i'(w) = \frac{1}{\pi i}\int_C g_i(z)\frac{dz}{(z-w)^2}, \tag{21}$$

and therefrom

$$\sum_{i=1}^{n} F_i'^2(w) = -\frac{1}{\pi^2}\int_C \int_C \sum_{i=1}^{n} g_i(z)g_i(\zeta)\frac{dzd\zeta}{(z-w)^2(\zeta-w)^2},$$

whence

$$\sum_{i=1}^{n} F_i'^2(w) = \frac{1}{2\pi^2}\int_C\int_C \sum_{i=1}^{n}[g_i(z)-g_i(\zeta)]^2\frac{dzd\zeta}{(z-w)^2(\zeta-w)^2}, \tag{22}$$

where the terms $g_i(z)^2$ and $g_i(\zeta)^2$ in the expansion of the bracket contribute zero, since

$$\int_C \frac{d\zeta}{(\zeta-w)^2} = 0, \quad \int_C \frac{dz}{(z-w)^2} = 0.$$

Next we observe that (using the subscript zero to denote the conjugate complex quantity)

$$\frac{1}{z(z-w)} \quad \text{and} \quad \frac{1}{z_0(z_0-w_0)}$$

are conjugate complex; consequently

$$\lambda\left\{\frac{1}{z(z-w)}-\frac{1}{z_0(z_0-w_0)}\right\} \text{ and } i\lambda\left\{\frac{1}{z(z-w)}+\frac{1}{z_0(z_0-w_0)}\right\}$$

are pure imaginary, where λ denotes a real parameter and w any fixed point interior to the unit circle. Hence the transformations

$$z' = z \exp \lambda\left\{\frac{1}{z(z-w)}-\frac{1}{z_0(z_0-w_0)}\right\}, \tag{23}$$

$$z' = z \exp i\lambda\left\{\frac{1}{z(z-w)}+\frac{1}{z_0(z_0-w_0)}\right\} \tag{24}$$

convert points of C into points of C, since they leave $|z|$ invariant. For $\lambda=0$ these transformations reduce to $z'=z$, or on C, $\theta'=\theta$, for which $d\theta'/d\theta=1$. Therefore, for λ small enough absolutely, $d\theta'/d\theta$ certainly remains positive since it is nearly equal to one; consequently the transformations (23) and (24) are, for $|\lambda|$ sufficiently small, monotonic continuous transformations of the unit circumference into itself. Accordingly, if we apply (23), together with the same transformation $\zeta\to\zeta'$, to $A(g^*)$, we get a function $A_1(\lambda)$ with a relative minimum at $\lambda=0$.

The transformation (23) may be expanded in a convergent power series in λ:

$$z' = z+\lambda\left\{\frac{1}{z-w}-\frac{z}{z_0(z_0-w_0)}\right\}+\cdots, \tag{25}$$

from which results a convergent power series for the differential:

$$\frac{dz'd\zeta'}{(z'-\zeta')^2}=\frac{dzd\zeta}{(z-\zeta)^2}+\lambda\{\cdots\}+\cdots.$$

Multiplying this by

$$\sum_{i=1}^{n}[{}'g_i(z')-{}'g_i(\zeta')]^2=\sum_{i=1}^{n}[g_i(z)-g_i(\zeta)]^2,$$

and integrating term by term,† we get a power series expansion for $A_1(\lambda)$ of the form $A_1(\lambda)=A(g^*)+\lambda V_1+\cdots$. Because of the minimum of $A_1(\lambda)$ at $\lambda=0$, we deduce

$$V_1 = 0. \tag{26}$$

Similarly, applying the transformation (24) in expanded form

$$z' = z + i\lambda\left\{\frac{1}{z-w}+\frac{z}{z_0(z_0-w_0)}\right\}+\cdots, \tag{27}$$

we get $A_2(\lambda)=A(g^*)+i\lambda V_2+\cdots$, and hence

$$V_2 = 0. \tag{28}$$

Now it is evident, as a formal matter, that we would get the same first power coefficients, or variations V_1, V_2, if we broke off (25) and (27) after the terms written. It is clear also that if we use any linear combination of the coefficients in (25) and (27), such as one-half the sum:

$$z' = z + \frac{\lambda}{z-w}, \tag{29}$$

and apply this transformation to $A(g^*)$, thus getting an expansion

$$A(g^*)+\lambda V_3+\cdots, \tag{30}$$

then V_3 is the same linear combination of V_1 and V_2:

$$V_3 = \tfrac{1}{2}(V_1+V_2);$$

consequently, by (26) and (28),

$$V_3 = 0. \tag{31}$$

We therefore apply the simple variation (29) to $A(g^*)$ and calculate V_3. The following are the computations, *all terms beyond the first power of* λ *being neglected*:

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† The validity of this operation is easily established by classic convergence theorems (Cauchy appraisal theorem, Weierstrass double-series theorem); see [1], §§13, 14.

$$z' = z + \frac{\lambda}{z - w}, \qquad \zeta' = \zeta + \frac{\lambda}{\zeta - w},$$

$$dz' = \left\{1 - \frac{\lambda}{(z - w)^2}\right\} dz, \quad d\zeta' = \left\{1 - \frac{\lambda}{(\zeta - w)^2}\right\} d\zeta,$$

$$dz'd\zeta' = \left\{1 - \frac{\lambda}{(z - w)^2} - \frac{\lambda}{(\zeta - w)^2}\right\} dzd\zeta,$$

$$z' - \zeta' = (z - \zeta)\left\{1 - \frac{\lambda}{(z - w)(\zeta - w)}\right\},$$

$$\frac{dz'd\zeta'}{(z' - \zeta')^2} = \frac{dzd\zeta}{(z - \zeta)^2}\left\{1 - \lambda\left[\frac{1}{(z - w)} - \frac{1}{(\zeta - w)}\right]^2\right\}.$$

The coefficient of λ is

$$\left(\frac{1}{z - w} - \frac{1}{\zeta - w}\right)^2 = \frac{(z - \zeta)^2}{(z - w)^2(\zeta - w)^2};$$

hence

$$\frac{dz'd\zeta'}{(z' - \zeta')^2} = \frac{dzd\zeta}{(z - \zeta)^2} - \lambda \frac{dzd\zeta}{(z - w)^2(\zeta - w)^2}.$$

Multiplying this by

$$\sum_{i=1}^{n} [{}'g_i^*(z') - {}'g_i^*(\zeta')]^2 = \sum_{i=1}^{n} [g_i^*(z) - g_i^*(\zeta)]^2,$$

and integrating, we get a power series expansion in λ, beginning

$$\frac{1}{4\pi} \int_C \int_C \sum_{i=1}^{n} [{}'g_i^*(z') - {}'g_i^*(\zeta')]^2 \frac{dz'd\zeta'}{(z' - \zeta')^2}$$

$$= A(g^*) - \lambda \cdot \frac{1}{4\pi} \int_C \int_C \sum_{i=1}^{n} [g_i^*(z) - g_i^*(\zeta)]^2 \frac{dzd\zeta}{(z - w)^2(\zeta - w)^2}.$$

Comparing with (30), we have therefore, by (31),

$$(32) \qquad \int_C \int_C \sum_{i=1}^{n} [g_i^*(z) - g_i^*(\zeta)]^2 \frac{dzd\zeta}{(z - w)^2(\zeta - w)^2} = 0.$$

By (22), this is the same as

$$\sum_{i=1}^{n} F_i'^2(w) = 0, \tag{33}$$

which proves that the surface $x_i = \Re F_i(w)$ determined by g^* is minimal.

EXCLUSION OF IMPROPER REPRESENTATIONS OF THE SECOND KIND. To show now that g^* cannot be constant on a partial arc of C, we shall first introduce into (33) the factor w^2, by writing $\sum_{i=1}^{n} w^2 F_i'^2(w) = 0$, and set†

$$wF_i'(w) = \Re wF_i'(w) + i\Im wF_i'(w);$$

and we shall then take the real part of the resulting equation, getting

$$\sum_{i=1}^{n} \{\Re wF_i'(w)\}^2 = \sum_{i=1}^{n} \{\Im wF_i'(w)\}^2. \tag{34}$$

Multiplying (21) by w and then expressing in terms of polar coordinates, we have $w = \rho e^{i\alpha}$, $z = e^{i\theta}$; and, separating out the imaginary part, we find

$$\Im wF_i'(w) = -\frac{1}{2\pi}\int_C \frac{2\rho(1-\rho^2)\sin(\theta-\alpha)}{[1-2\rho\cos(\theta-\alpha)+\rho^2]^2} g_i^*(\theta)d\theta. \tag{35}$$

According to a result of Fatou [30], the limit of this harmonic function as w approaches a boundary point $e^{i\theta}$ exists as a unique value and is equal to the derivative $dg^*(\theta)/d\theta$ at every point where this derivative exists and is continuous.

Since, by the hypothesis which we are trying to disprove, $g^*(\theta)$ is constant on a partial arc of C, it follows that $\Im wF_i'(w) = 0$ on this arc, for $i = 1, 2, \cdots, n$. Hence, by (34), $\Re wF_i'(w) = 0$, also for $i = 1, 2, \cdots, n$, on the same arc. Consequently $wF_i'(w) = 0$ on the arc, for $i = 1, 2, \cdots, n$. This enables us to conclude that $wF_i'(w) \equiv 0$, $(i = 1, 2, \cdots, n)$, since, by Schwarzian symmetry, involving inversion in C, the arc can be made interior to a domain of regularity of the function.

Thus $F_i(w)$ would reduce to a constant, and consequently also g_i^*, the boundary value of $\Re F_i(w)$. But we have assumed that g^* is constant only on a *partial* arc of C, the possibility of

† Of course, there should be no confusion of i, the index that runs from 1 to n, with i, the square root of -1.

a degenerate representation having been previously ruled out by the condition of three distinct fixed points on Γ and C.

The contradiction thus exhibited completes the reasoning. The minimizing representation g^* of $A(g)$ is proper, and the minimal surface determined by it is therefore bounded by Γ. This solves the Plateau problem for any contour Γ with finite $m(\Gamma)$.

ARBITRARY JORDAN CONTOUR. To treat the case $m(\Gamma) = +\infty$, we represent the contour Γ as limit of a sequence of contours $\bar{\Gamma}$ with finite $m(\bar{\Gamma})$, for instance, polygons. Let $x_i = \bar{f}_i(t)$ be the equations of $\bar{\Gamma}$, $x_i = f_i(t)$ of Γ; then $\bar{f}_i(t) \to f_i(t)$. Each contour $\bar{\Gamma}$ has a parameter θ solving the Plateau problem for $\bar{\Gamma}$, θ being derived from t by a monotonic continuous transformation $t = \bar{\mu}(\theta)$. The sequence of monotonic transformations $\bar{\mu}$, by the compactness of the set of all such transformations, contains a convergent subsequence, with limit a proper or improper monotonic transformation $t = \mu(\theta)$. We thus obtain a certain representation of Γ in the form

$$x_i = g_i(\theta) = f_i(\mu(\theta)),$$

where $g_i(\theta)$ is the limit of a sequence of Plateau representations of $\bar{\Gamma}$,

$$x_i = \bar{g}_i(\theta) = \bar{f}_i(\bar{\mu}(\theta)).$$

If $x_i = \Re F_i(w)$ and $\bar{x}_i = \Re \bar{F}_i(w)$ are formed respectively from g, $\bar{g}$ by the Poisson formula (18), then evidently, with $F'(w)$, $\bar{F}'(w)$ given by (21), we have $\bar{F}_i'(w) \to F_i'(w)$, for the integrand remains uniformly bounded for any fixed value of w. Since during the limit process we have always $\sum_{i=1}^{n} \bar{F}_i'^2(w) = 0$, it follows that

$$\sum_{i=1}^{n} F_i'^2(w) = 0. \tag{36}$$

It remains to prove that $x_i = g_i(\theta)$ is a proper representation of Γ. The condition (36) having been established, the reasoning gone through just previously, based on Fatou's theorem and Schwarzian symmetry, applies to rule out improper representations of the second kind. The possibility of a degenerate representation is excluded by restricting the monotonic transformations $\bar{\mu}$ to leave fixed three chosen distinct points of the unit

circumference, as can be done without harm to generality by use of a linear fractional transformation of the unit circle into itself, this involving three arbitrary constants.

However, the possibility of an improper representation of the first kind cannot now be excluded by the argument given for the case of finite $m(\Gamma)$: that then $A(g) = +\infty$, for here $A(g)$ is identically $+\infty$. To rule out this type of improper representation, we now show that it is inconsistent with (36).

In an improper representation of the first kind there is a point P on the unit circumference where the vector function g has unequal one-sided limits, the distinct points (a_i) and (b_i) of Γ. Without loss of generality we may rotate the unit circle till P is at $w = 1$.

We next invert the unit circle with respect to an orthogonal circle whose center O is on the radius to P prolonged.† This inversion, combined with a reflection in the real axis, gives a transformation

$$w' = \frac{\alpha w + \beta}{\gamma w + \delta}, \tag{37}$$

which converts the unit circle into itself, and converts the functions $F_i(w)$ into new ones which evidently still obey the relation (36).

If now O is made to tend to P, it is seen that the transformation (37) tends to spread the values of g just below P over the whole upper semi-circumference and the values just above P over the whole lower semi-circumference, so that g_i tends to the function equal to a_i on the upper semi-circumference and to b_i on the lower.

Consequently, by formula (13), $F_i(w)$ tends to the function

$$\begin{aligned} G_i(w) &= \frac{1}{2\pi}\int_0^{\pi} a_i \frac{e^{i\theta} + w}{e^{i\theta} - w}\, d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} b_i \frac{e^{i\theta} + w}{e^{i\theta} - w}\, d\theta \\ &= \frac{a_i - b_i}{\pi i} \log i \frac{1 + w}{1 - w}. \end{aligned} \tag{38}$$

At the same time $F_i'(w)$ tends to $G_i'(w)$, and $\sum_{i=1}^{n} F_i'^2(w)$ to $\sum_{i=1}^{n} G_i'^2(w)$. But, as calculated from (38),

† See [11], p. 305.

$$(39) \qquad \sum_{i=1}^{n} G_i'^2(w) = -\frac{4}{\pi^2} \sum_{i=1}^{n} (a_i - b_i)^2 \cdot \frac{1}{(1-w^2)^2},$$

which is $\neq 0$, since the points (a_i) and (b_i) are distinct.† Thus we have contradiction with the fact that in the limit process in which $O \to P$ the condition $\sum_{i=1}^{n} F_i'^2(w) = 0$ is always obeyed.

LEAST AREA. Having taken the Weierstrass formulas as definition of minimal surface, we have not concerned ourselves in the preceding account with the least area property of the minimal surface M whose existence has now been established. It is, however, a fact that M has the minimum area of any simply-connected surface bounded by Γ [11, §26; 15].

This statement is significant only in case the minimum area $m(\Gamma)$ is finite. But even when $m(\Gamma) = +\infty$, there is still a good sense in which the least area property is had by the surface M. Let Γ' denote any closed contour on M having no point in common with the boundary Γ. Then the area enclosed by Γ' on M is finite and an absolute minimum with respect to all simply-connected surfaces bounded by Γ' [14]. That is, the infinite part of the area of M lies, so to say, altogether on its boundary. If Γ_ρ denote the contour on M corresponding to the circle of radius $\rho < 1$ concentric with the unit circle, then as $\rho \to 1$ the area enclosed on M by Γ_ρ becomes infinite in at most the order of $(1-\rho)^{-3}$; in fact, if A_ρ denote the area intercepted on M by Γ_ρ, then, for all $\rho < 1$, we have

$$(40) \qquad A_\rho \leqq \frac{3\pi D^2}{(1-\rho)^3},$$

where D denotes the length of the diagonal of any rectangular parallelepiped containing the contour Γ.

6. *Conformal Mapping.* To get the Riemann mapping theorem for Jordan regions together with the Osgood-Carathéodory theorem concerning the behavior of the map on the boundaries is merely a matter of putting $n = 2$ in the preceding work.

Let the plane Jordan curve Γ bound a region R. Then, by the foregoing theory, there exists a topological correspondence between Γ and the unit circumference C that gives a parametric representation of Γ of the form

† Here again, use is being made of the condition that the contour has no double points.

$$x_1 = g_1^*(z), \quad x_2 = g_2^*(z), \tag{41}$$

such that the functions

$$\begin{aligned} F_1(w) &= \frac{1}{2\pi i}\int_C g_1^*(z)\,\frac{z+w}{z-w}\,\frac{dz}{z}, \\ F_2(w) &= \frac{1}{2\pi i}\int_C g_2^*(z)\,\frac{z+w}{z-w}\,\frac{dz}{z} \end{aligned} \tag{42}$$

obey the condition

$$F_1'^2(w) + F_2'^2(w) = 0, \text{ or } F_1'(w) \pm iF_2'(w) = 0.$$

We have seen in §1 that this implies that the functions

$$x_1 = \Re F_1(w), \quad x_2 = \Re F_2(w) \tag{43}$$

are conjugate harmonic, or that

$$W = x_1 + ix_2 = F(w) \tag{44}$$

is a monogenic function of w. As such, it is therefore expressible in terms of its boundary values on C in the form

$$g^*(z) = g_1^*(z) + ig_2^*(z)$$

by the Cauchy integral formula

$$F(w) = \frac{1}{2\pi i}\int_C \frac{g^*(z)dz}{z-w}. \tag{45}$$

The equations (44) and (45) are merely combined forms of (43) and (42).

Because the transformation (44) converts the unit circumference in a one-to-one continuous way into Γ, it follows, by an important theorem of Darboux,† that it converts the interior of the unit circle into the interior R of Γ in a one-to-one manner and conformally, without singular points. The mapping theorem of Riemann is proved.

To obtain now the theorem of Osgood and Carathéodory requires only the simple observation that the formulas (42) and (43) which define the conformal map are equivalent to Poisson's integral, which has the well known property of attaching

† Osgood, *Funktionentheorie*, 1923, p. 377.

continuously to its boundary values $g_1^*(z)$ and $g_2^*(z)$, if these are continuous, as they are.

For any contour Γ, it is easy to show that the minimum value of $A(g)$ is equal to the area of the minimal surface M determined by g^*. Therefore, in the case of a plane Jordan curve bounding a region R,

$$\min A(g) = \text{inner area of } R. \tag{46}$$

The inner area means the upper bound of the areas of all polygons which, with their boundaries, are interior to R.† This remark made, we can now summarize the results of this section in the following proposition.

Let Γ denote any plane Jordan curve bounding a region R, and let $Z=g(z)$ denote an arbitrary representation of Γ, locus of Z, as topological image of the unit circumference C, locus of z. The range of values of the functional

$$A(g) = \frac{1}{4\pi}\int_C\int_C |\, g(z) - g(\zeta)\,|^2 \frac{dz\,d\zeta}{(z-\zeta)^2}$$

when all representations g are considered will consist exactly of all positive real numbers $\geqq$ the inner area of R. This minimum value is attained for a certain representation $Z=g^*(z)$ (determined up to linear fractional transformation of C into itself). Then the transformation $w \to W$ defined by the integral formula of Cauchy,

$$W = \frac{1}{2\pi i}\int_C \frac{g^*(z)dz}{z-w},$$

establishes a one-to-one conformal map of the interior of the unit circle on the region R, and this conformal map attaches continuously to the topological correspondence g^* between Γ and C.

7. *Dirichlet Principle.* The implications for the Dirichlet problem of the results just stated are apparent. The Dirichlet problem for any continuous distribution of assigned values on

† To be distinguished from the outer area, lower bound of the areas of all simply-connected polygons whose boundaries are exterior to R. The first example of a Jordan curve with inner area less than outer area, or where the curve itself has a positive area, was given by Osgood, Transactions of this Society, vol. 4 (1903), pp. 107–112.

any Jordan curve is reduced to the same problem for the circle, and therefore solved immediately by Poisson's integral.

It will be observed that the proof of the Dirichlet principle thus given consists in introducing instead of the Dirichlet functional $D(\phi)$, against which the Weierstrass objection applies (the range of the argument $[\phi]$ is not compact), the functional $A(g)$ which, on the contrary, is amenable to immediate treatment according to the Weierstrass-Fréchet pattern.

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PSEUDO-COVARIANTS OF AN n-IC IN m VARIABLES IN A GALOIS FIELD THAT CONSIST OF TERMS OF THIS n-IC*

BY A. D. CAMPBELL

Let us consider the n-ic in m variables

$$(1) \qquad f(x_1, x_2, \cdots, x_m) = \sum a_{i,j,\cdots,k} x_1^i x_2^j \cdots x_m^k = 0,$$

where $i, j, \cdots, k = 0, 1, 2, \cdots, n$, and $i+j+\cdots+k=n$; also where the coefficients and variables represent numbers in a Galois field of order p^n. If we subject (1) to the transformation

$$(2) \qquad \rho x_\lambda = \sum \alpha_{\lambda\mu} x_\mu' ,$$

where $\lambda, \mu = 1, 2, \cdots, m$, and the coefficients and variables represent numbers in the same Galois field, we obtain

$$(3) \qquad \begin{aligned} f'(x_1', x_2', \cdots, x_m') &= \sum a_{i,j,\cdots,k} (\alpha_{11} x_1' + \cdots + \alpha_{1m} x_m')^i \\ &\cdot (\alpha_{21} x_1' + \cdots + \alpha_{2m} x_m')^j \cdots (\alpha_{m1} x_1' + \cdots + \alpha_{mm} x_m')^k \\ &= \sum a'_{i,j,\cdots,k} x_1'^i x_2'^j \cdots x_m'^k = 0. \end{aligned}$$

We note from (3) that the only terms in (1) that can furnish terms in the primed variables whose factors all have exponents less than p (the prime modulus of this Galois field) are those terms in (1) whose factors all have exponents less than p. Therefore if such terms are present in (1), the n-ic has what we shall call a pseudo-covariant consisting of the above-mentioned terms of (1), that has the form

$$(4) \qquad \sum a_{r,s,\cdots,t} x_1^r x_2^s \cdots x_m^t ,$$

where $r+s+\cdots+t=n$ and $r<p$, $s<p$, $\cdots$, $t<p$. The set of terms (4) contribute also other types of primed terms to (3) after we use (2) on (1), but no other terms in (1) except the set (4) furnish (3) with primed terms of the same type as the unprimed terms of (4).

If (1) has no terms of the form (4), we note that only the terms in (1) having each just one factor of the form x_i^u, where

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$p \leqq u < p^2$, can contribute to (3) primed terms of this same form. Therefore, if (1) has no pseudo-covariant (4), it has the pseudo-covariant

$$\sum a_{u,s,\cdots,t} x_1^u x_2^s \cdots x_m^t, \tag{5}$$

where $u+s+\cdots+t=n$ and just one of the exponents u, s, $\cdots$, t in each term lies between p and p^2, while the other exponents are less than p.

Next if (1) has no terms of the form (5), then the n-ic has the pseudo-covariant

$$\sum a_{u,v,\cdots,t} x_1^u x_2^v \cdots x_m^t, \tag{6}$$

where $u+v+\cdots+t=n$ and just two of the exponents u, v, $\cdots$, t in each term lie between p and p^2, while the other exponents are less than p.

If (1) has no terms each with just two exponents between p and p^2 and their other exponents less than p, we find a pseudo-covariant whose terms have each just three exponents lying between p and p^2 while their other exponents are less than p.

If (1) has no terms with all their exponents less than p^2, the n-ic has a pseudo-covariant consisting of all the terms in (1) that have each just one factor whose exponent lies between p^2 and p^3 while the other exponents are less than p^2. If no such pseudo-covariant exists then the n-ic has a pseudo-covariant consisting of all the terms in (1) that have each just two factors whose exponents lie between p^2 and p^3 while their other exponents are less than p^2. Similarly we can generalize to the cases of pseudo-covariants consisting of all the terms of (1) having each a certain number of exponents lying between p^3 and p^4 but their other exponents less than p^3, between p^4 and p^5, and so forth.

Let us consider next the relations between the coefficients of (3) and the coefficients of (1), namely,

$$\sigma a'_{i,j,\cdots,k} = \sum \beta_{\lambda,\mu,\cdots,\nu} a_{\lambda,\mu,\cdots,\nu}, \tag{7}$$

where $i, j, \cdots, k$ and $\lambda, \mu, \cdots, \nu$ have the values $0, 1, 2, \cdots, n$. This is the transformation between the coefficients of (1) and (3) that is caused by the transformation (2) upon the variables x_i. Capelli has proved (*Lezioni*, p. 202) that any linear transformation on m variables can be obtained by the successive ap-

plication of a finite number of linear transformations of the following types:

$$(8)\qquad \begin{aligned} T^a &: x_1 = x_1', \cdots, x_{l-1} = x_{l-1}', x_l = ax_l', \\ &x_{l+1} = x_{l+1}', \cdots, x_m = x_m', \end{aligned}$$

and

$$\begin{aligned} T_{lk} &: x_1 = x_1', \cdots, x_{l-1} = x_{l-1}', x_l = x_l' + x_k', \\ &x_{l+1} = x_{l+1}', \cdots, x_m = x_m', \end{aligned}$$

or by T_1^a, T_{12} supplemented by transformations of the type

$$x_1 = x_j', x_j = x_1', x_k = x_k', \quad (k \neq 1, k \neq j).$$

We note that a must equal the determinant of the transformation (2) that is analyzed into a product of transformations of the form (8). If we subject (1) to T_1^a, we see that the determinant of (7) is a power of the determinant of (2).

To find what power of a the determinant a' of (7) is we arrange (1) in ascending powers of x_1. We note that the number of terms in a homogeneous form of degree k in l variables is

$$(9)\qquad {}_kS_l = \frac{(k+1)(k+2)(k+3)\cdots(k+l-1)}{(l-1)!}.$$

We subject (1) to T_1^a. We note that the coefficient $a'_{i,j,\cdots,k}$ in (7) resulting from T_1^a has the value $a^i a_{i,j,\cdots,k}$. There are ${}_{n-1}S_{m-1}$ similar coefficients in (3) for any given i. From this we see that the determinant of (7) is

$$(10)\qquad a' = a^{{}_np_m},$$

where

$${}_np_m = 0\ {}_nS_{m-1} + 1\ {}_{n-1}S_{m-1} + 2\ {}_{n-2}S_{m-1} + \cdots + n\ {}_0S_{m-1}.$$

Next we see that the coefficients of each of the several pseudo-covariants are transformed by a transformation whose determinant a'' is also a power of the determinant a of (2). To find what power a'' is of a, we suppose the highest exponent of any factor in any term to be s. We arrange the pseudo-covariant in descending powers of x_1. We subject (1) to T_1^a. The coefficient $a'_{s,j,\cdots,k}$ has the value $a^s a_{s,j,\cdots,k}$. We count the number of partitions of $n-s$ into sums of numbers less than or equal to s. If the terms of the pseudo-covariant have each just one factor with exponent equal to s (where $p \leqq s < p^2$, say) we take only

the partitions of $n-s$ into the sums of numbers less than s. A similar remark applies to each of the other types of pseudo-covariants.

We treat the case with $p \leqq s < p^2$ and with only one such exponent s per term. The generalization to the other cases is evident. To each of these partitions we affix the number of permutations of $m-1$ things taken l at a time, where l is the number of terms in the partition, allowing for repetitions. The sum of all these affixed numbers we label Π_1. By this means we count all the terms in the pseudo-covariant each of which has x_1^s as a factor.

Next we consider the coefficients $a'_{s-1,j,\cdots,k}$. We take the number of partitions of $n-s-(s-1)$ into sums of numbers less than s. We add up all the permutations of $m-2$ things taken l at a time associated with these partitions as above and we call this sum Π_2. We multiply Π_2 by $m-1$ because the exponent s may be placed on any one of the variables $x_2, x_3, \cdots, x_m$. Next we consider the coefficients $a'_{s-2,j,\cdots,k}$ and the partitions of $n-s-(s-2)$ with the associated number Π_3. We must multiply Π_3 by $m-1$.

From the above discussion we see that the transformation of the coefficients of the pseudo-covariant has the determinant

$$a'' = a^{\Pi}, \tag{11}$$

where

$$\Pi = \Pi_1 s + (m-1)[\Pi_2(s-1) + \Pi_3(s-2) + \cdots + \Pi_s].$$

An interesting special case of the above is found when $p=2$ and the n-ic (1) has terms consisting entirely of linear factors. This can happen only when $m \geqq n$. In this case the above mentioned determinant is a so-called compound determinant of the determinant of (2). Compare Metzler-Muir, *Determinants*, page 178. In this case we have $a'' = a^{m-1C_{n-1}}$, where ${}_{m-1}C_{n-1}$ is the number of combinations of $m-1$ things taken $n-1$ at a time. If $m=n$ we have $a''=a$.

The next question that arises is as to the form of the transformation (7). We note that the general case is given in a previous paper.* We give now a method of obtaining (7) and also of

* *Note on linear transformations of n-ics in m variables*, this Bulletin, vol. 35 (1929), pp. 691–694.

obtaining the transformations of the coefficients of the several pseudo-covariants. We note the multinomial expansion

$$(12) \qquad (a+b+\cdots+k)^n = \sum \frac{n!}{\alpha!\beta!\cdots\kappa!} a^\alpha b^\beta \cdots k^\kappa,$$

where $\alpha+\beta+\cdots+\kappa=n$.

We shall illustrate the method by a particular case. The generalization will then be evident. Suppose we have a term $a_{23}x_1^2x_2^3$ of a quintic in three variables and we wish to find the coefficient a'_{221} of the term in $x_1'^2x_2'^2x_3'$ that (2) yields us from this term. We put

$$y_1 = \alpha_{11}x_1', \quad y_2 = \alpha_{12}x_2', \quad \cdots, \quad z_1 = \alpha_{21}x_1', \quad \cdots, \quad z_3 = \alpha_{23}x_3',$$

and we have

$$a_{23}(y_1+y_2+y_3)^2(z_1+z_2+z_3)^3$$
$$= a_{23}\left(\sum \frac{2!}{\alpha_1!\beta_1!\gamma_1!} y_1^{\alpha_1}y_2^{\beta_1}y_3^{\gamma_1}\right)\left(\sum \frac{3!}{\alpha_2!\beta_2!\gamma_2!} z_1^{\alpha_2}z_2^{\beta_2}z_3^{\gamma_2}\right).$$

To obtain a'_{221} we multiply a_{23} by the sum s of all the products of powers of $\alpha_{11}, \alpha_{12}, \cdots$ obtained from the above parentheses when we set

$$\alpha_1+\alpha_2=2, \qquad \beta_1+\beta_2=2, \qquad \gamma_1+\gamma_2=1,$$
$$\alpha_1+\beta_1+\gamma_1=2, \qquad \alpha_2+\beta_2+\gamma_2=3.$$

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A NOTE ON THE EQUIVALENCE OF ALGEBRAS OF DEGREE TWO*

BY A. A. ALBERT

The simplest type of normal simple algebra over any non-modular field F is the cyclic algebra of degree two (order four)

$$(1)\quad Q(\alpha, \beta) = (1, i, j, ij),\ ji = -ij,\ i^2 = \alpha \neq 0,\ j^2 = \beta \neq 0,$$

(α and β in F), the so-called generalized quaternion algebra over F. Of great importance in the theory of linear algebras is the problem of finding conditions that two given normal simple algebras of the same degree shall be equivalent. But this problem has not, as yet, been explicitly solved even for the above simplest case of algebras of degree two except when F is an algebraic field.† The purpose of this brief note is to give a simplification of my own previous results for rational algebras of degree two and thereby to give simple explicit conditions that any two generalized quaternion algebras over any non-modular field F shall be equivalent.

We consider an algebra $Q(\alpha, \beta)$. A quantity x in $Q(\alpha, \beta)$ but not in F has the property $x^2 = \gamma$ in F if and only if

$$(2)\quad x = \xi_1 i + (\xi_2 + \xi_3 i)j,\ x^2 = \gamma = \xi_1^2\alpha + \xi_2^2\beta - \xi_3^2\alpha\beta.$$

Suppose first that another algebra $Q(\gamma, \delta)$ has the property $\gamma = \xi_1^2\alpha$ for ξ_1 in F. Then, as is well known,‡ we have the following lemma.

LEMMA. *If $\gamma = \xi_1^2\alpha$, then $Q(\alpha, \beta)$ is equivalent to $Q(\gamma, \delta)$ if and only if*

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* Presented to the Society, February 25, 1933.

† See this Bulletin, vol. 36 (1930), pp. 535–540, for algebras of degree two over any algebraic number field, and Hasse's arithmetic invariant theory in the Transactions of this Society, vol. 34 (1932), for algebras of degree n over any algebraic number field. Hasse's conditions of course have no meaning for the case we are considering.

‡ The case $n = 2$ of Hasse's Theorem (2.12) (loc. cit. p. 173) if $\alpha \neq \epsilon^2$ for any ϵ of F. If $\alpha = \epsilon^2$, then both of the above algebras are total matric algebras and are equivalent for any β and δ. But also the equation $\delta = (\xi_4^2 - \xi_5^2\alpha)\beta \neq 0$ has the solution $2\xi_4 = (1 + \delta\beta^{-1})$, $2\xi_5\epsilon = (1 - \delta\beta^{-1})$.

$$\delta = (\xi_4^2 - \xi_5^2\alpha)\beta,$$

for ξ_4 and ξ_5 in F.

Let next $Q(\alpha, \beta)$ and $Q(\gamma, \delta)$ be equivalent. Since then there must exist an x in $Q(\alpha,\beta)$ for which $x^2=\gamma$, we must have equation $(2)_2$ for ξ_1, ξ_2, ξ_3 not all zero and in F. If $\xi_2=\xi_3=0$, then our equivalence conditions are given in the Lemma above. Otherwise

$$y = \xi_3\alpha j + \xi_2 ij \neq 0, \tag{3}$$

and I have proved* that

$$Q(\alpha, \beta) = (1, x, y, xy) = Q(\alpha_0, \beta_0), \tag{4}$$

where $yx=-xy$, $x^2=\alpha_0=\gamma$, $y^2=\beta_0=-\alpha\beta(\xi_2^2-\xi_3^2\alpha)$. By the Lemma above we must have $\delta=(\xi_4^2-\xi_5^2\gamma)\beta_0$ for ξ_4, ξ_5 in F. Combining this result with the Lemma we have the following result.

THEOREM. *A necessary condition that two normal simple algebras of degree two over any non-modular field F,*

$$A = Q(\alpha, \beta), B = Q(\gamma, \delta),$$

be equivalent is that there exist ξ_1, ξ_2, ξ_3 in F for which

$$\gamma = \xi_1^2\alpha + \xi_2^2\beta - \xi_3^2\alpha\beta \neq 0. \tag{5}$$

For any ξ_1, ξ_2, ξ_3 satisfying (5) *the algebras A and B are equivalent if and only if*

$$\delta = (\xi_4^2 - \xi_5^2\gamma)\beta_0, \tag{6}$$

with ξ_4 and ξ_5 in F, where

$$\beta_0 = \beta, \text{ or } \beta_0 = -\alpha\beta(\xi_2^2 - \xi_3^2\alpha), \tag{7}$$

according as ξ_2 and ξ_3 are or are not both zero.

We have therefore proved that all algebras $Q(\gamma, \delta)$ equivalent to a given A are obtained by (5), (6), (7) as $\xi_1, \cdots, \xi_5$ range over all quantities of F for which $\gamma\delta\neq 0$.

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* A rational proof holding for any F was given in my Bulletin paper (loc. cit.). This result was also used to prove the above lemma but by a rather complicated computation.

A NOTE ON BILINEAR FORMS

BY R. E. A. C. PALEY†

1. *Introduction.* In this note I consider the proof of the following theorem due to M. Riesz.‡

THEOREM. *Let* $M^*(\alpha, \gamma)$ *denote the maximum for variable* x *of*

$$(1) \qquad \frac{\left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\gamma} \right)^{\gamma}}{\left(\sum_{m=1}^{M} C_m |x_m|^{1/\alpha} \right)^{\alpha}},$$

where the numbers A_{mn}, $(1 \leqq m \leqq M, 1 \leqq n \leqq N)$, *are fixed and the numbers* B_n, $(1 \leqq n \leqq N)$, *and* C_m, $(1 \leqq m \leqq M)$, *are all positive. Then* $\log M^*(\alpha, \gamma)$ *is a convex function of the variables* (α, γ) *in the triangle*

$$0 \leqq \gamma \leqq \alpha \leqq 1$$

of the (α, γ) *plane.*

I show here that it is sufficient to prove the theorem in the case where $\gamma = \alpha$, $(0 \leqq \alpha \leqq 1)$. Suppose the theorem to have been established in this case.

2. *Proof of the Theorem.* Let d_n, $(1 \leqq n \leqq N)$, be a set of positive numbers for which

$$(2) \qquad \sum_{n=1}^{N} B_n d_n = 1.$$

Then, if $\gamma \leqq \alpha$ the numerator of (1) is the maximum, for all sets of d_n satisfying (2), of the expression

$$(3) \qquad \left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\alpha} d_n^{1-\gamma/\alpha} \right)^{\alpha}.$$

† With profound regret, the Editors note that the author of this paper died on April 3, 1933. This paper therefore appears posthumously, though the author had read the proofs of it.

‡ M. Riesz, *Sur les maxima des formes bilinéaires, et sur les fonctions linéaires*, Acta Mathematica, vol. 49 (1926), pp. 465–497.

Now on any segment of a line contained strictly in the triangle $0 \leqq \gamma \leqq \alpha \leqq 1$, $1-\gamma/\alpha$ is of the form $\lambda+\mu/\alpha$, where λ and μ are constants. Substituting in (3), we see that the maximum of (1) is the maximum, for varying x and d, of the expression

$$(4) \qquad \left(\frac{\sum_{n=1}^{N} B_n' \left| \sum_{m=1}^{M} A_{mn}' x_m \right|^{1/\alpha}}{\sum_{m=1}^{M} C_m |x_m|^{1/\alpha}} \right)^{\alpha},$$

where A_{mn}', B_n' vanish whenever $d_n = 0$, and otherwise

$$A_{mn}' = A_{mn} d_n^{\mu}, \quad B_n' = B_n d_n^{\lambda}.$$

Now assuming the theorem to have been already established in the case where $\gamma = \alpha$, we see that, for each fixed set of d_n the logarithm of the maximum of the expression (4) is convex, and hence $\log M^*(\alpha, \gamma)$, as the maximum of a set of convex functions, is also convex. This proves the required result.

3. *A Correction.* I take this opportunity of remarking that a previous paper† of mine, which gives a proof of the theorem of this note, contains an error, which has been pointed out to me independently by Professor Pólya, Professor M. Riesz, and Professor Tamarkin. In fact, in §2 of this paper, I assume, quite unjustifiably, that $\log M^*(\alpha, \gamma)$ is differentiable, while all that we can legitimately assert is its continuity. The argument can be rectified by some complicated, but rather uninteresting, lemmas on convex functions, but I shall not attempt to carry this out here.

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† R. E. A. C. Paley, *A proof of a theorem on bilinear forms*, Journal of the London Mathematical Society, vol. 6 (1931), pp. 226–230.

ON ϑ, ϕ IDENTITIES

BY W. H. GAGE

1. *Introduction.* In a recent series of papers* Bell has referred to several ϑ, ϕ identities which were derived at his suggestion by the writer. The purpose of this paper is to indicate the method of deriving these identities and to point out how they may be generalized.

2. *Formulas of Degree Three in Four Variables.* Consider the well known theta formula of Jacobi

$$(1)\qquad 2\vartheta_1(w)\vartheta_1(u) = \begin{vmatrix} \vartheta_0\left(\frac{w+u}{2}, q^{1/2}\right), & \vartheta_3\left(\frac{w+u}{2}, q^{1/2}\right) \\ \vartheta_0\left(\frac{w-u}{2}, q^{1/2}\right), & \vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right) \end{vmatrix}.$$

From (1) we easily prove that $2\vartheta_1(w)\vartheta_1(u)\vartheta_3((u+z)/2, q^{1/2})$ equals

$$(2)\qquad \begin{vmatrix} \vartheta_0\left(\frac{w+u}{2}, q^{1/2}\right)\vartheta_3\left(\frac{u+z}{2}, q^{1/2}\right) & \\ -\vartheta_3\left(\frac{w+u}{2}, q^{1/2}\right)\vartheta_0\left(\frac{u+z}{2}, q^{1/2}\right), & \vartheta_3\left(\frac{w+u}{2}, q^{1/2}\right) \\ \vartheta_0\left(\frac{w-u}{2}, q^{1/2}\right)\vartheta_3\left(\frac{u+z}{2}, q^{1/2}\right) & \\ -\vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right)\vartheta_0\left(\frac{u+z}{2}, q^{1/2}\right), & \vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right) \end{vmatrix}$$

$$= 2\begin{vmatrix} \vartheta_1\left(\frac{w+z+2u}{2}\right)\vartheta_1\left(\frac{w-z}{2}\right), & \vartheta_3\left(\frac{w+u}{2}, q^{1/2}\right) \\ \vartheta_1\left(\frac{w+z}{2}\right)\vartheta_1\left(\frac{w-z-2u}{2}\right), & \vartheta_3\left(\frac{w-u}{2}, q^{1/2}\right) \end{vmatrix}.$$

* E. T. Bell, *Quadratic Partitions*, this Bulletin, Paper I, vol. 37 (1931), pp. 871–875; Papers II, III, and IV, vol. 38 (1932), pp. 551–554, pp. 569–572, and pp. 697–699. For notations undefined in the present paper, the reader is referred to Bell's four papers.

Equating the two values of $\vartheta_1(w)$ obtained from (2) and from the corresponding expression for $2\vartheta_1(w)\vartheta_1(v)\vartheta_3((v-z)/2), q^{1/2})$, dividing the resulting equation by $\vartheta_1((w+z)/2)\vartheta_1((w-z)/2)$, and simplifying, we establish the ϑ, ϕ identity

$$
\begin{aligned}
&\phi_{111}\left(\frac{w+z}{2},\ u\right)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-u}{2},\ q^{1/2}\right)\\
&+\phi_{111}\left(\frac{w-z}{2},\ -u\right)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+u}{2},\ q^{1/2}\right)\\
&-\phi_{111}\left(\frac{w-z}{2},\ v\right)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-v}{2},\ q^{1/2}\right)\\
&-\phi_{111}\left(\frac{w+z}{2},\ -v\right)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+v}{2},\ q^{1/2}\right)\equiv 0.
\end{aligned}
\tag{3}
$$

From (1) other identities obviously may be derived. In (2), for example, we may introduce the factor $\vartheta_0((u+z)/2), q^{1/2})$ instead of $\vartheta_3((u+z/2), q^{1/2})$, and use a theta formula similar to but different from (1) to complete the simplification; or we may combine the factors with the second column of the determinant in (1) instead of with the first. The resulting identities all have the same arguments u, v, z, $(w+z)/2$, $(v-z)/2$, etc., but the subscripts on the ϕ's and the ϑ's are changed.

Formula (1) is only one source of such identities, for a large number of Jacobi's formulas similar to (1) yield sets. Not all the identities, however, are independent. If in (3), for example, we replace q by $-q$ or in turn increase each of the variables by multiples of $\pi/2$ or π, we can write down a chain of new identities (involving the same arguments) until finally the original identity is restored. Let us denote (3) by

$$(111, 33) + (111, 33) - (111, 33) - (111, 33) \equiv 0. \tag{4}$$

Replacing w by $w+\pi$ in (4), we get

$$(221, 30) + (221, 30) - (221, 30) - (221, 30) \equiv 0, \tag{5}$$

and increasing each of w, z, u, v in (4) by $\pi/2$ gives us

$$(122, 33) - (212, 30) + (212, 03) - (122, 00) \equiv 0. \tag{6}$$

3. *More General Cases.* Each of the identities (4), (5), and (6), exhibited above, is of degree three (one in ϕ and two in ϑ) and each contains four variables. In the following paragraphs we indicate how identities of greater degree or in more variables may be derived from any identity of type (4).

One method is to utilize well known relations such as the following:

$$
\begin{aligned}
\vartheta_0\vartheta_0(2a, q^2) &= \vartheta_0(a)\vartheta_3(a),\\
\vartheta_0\vartheta_1(2a, q^2) &= \vartheta_1(a)\vartheta_2(a),\\
\vartheta_2\vartheta_1(a, q^{1/2}) &= 2\vartheta_1(a)\vartheta_0(a),\\
\vartheta_2\vartheta_2(a, q^{1/2}) &= 2\vartheta_3(a)\vartheta_2(a),\\
\vartheta_2\vartheta_2(a, iq^{1/2}) &= 2e^{i\pi/4}\vartheta_2(a)\vartheta_0(a),\\
\vartheta_2\vartheta_1(a, iq^{1/2}) &= 2e^{i\pi/4}\vartheta_1(a)\vartheta_3(a).
\end{aligned}
\tag{7}
$$

By means of (7), a ϕ may be expressed as the product of two ϕ's and a theta function as the product of two ϑ's. If ϕ_{111} in (4) be so expressed, we obtain

$$
\begin{aligned}
&\phi_{111}\left(\frac{w+z}{2}, u\right)\phi_{222}\left(\frac{w+z}{2}, u\right)\vartheta_3(v-z)\vartheta_3(w-u)\\
&+\phi_{111}\left(\frac{w-z}{2}, -u\right)\phi_{222}\left(\frac{w-z}{2}, -u\right)\vartheta_3(v-z)\vartheta_3(w+u)\\
&-\phi_{111}\left(\frac{w-z}{2}, v\right)\phi_{222}\left(\frac{w-z}{2}, v\right)\vartheta_3(u+z)\vartheta_3(w-v)\\
&-\phi_{111}\left(\frac{w+z}{2}, -v\right)\phi_{222}\left(\frac{w+z}{2}, -v\right)\vartheta_3(u+z)\vartheta_3(w+v)\equiv 0,
\end{aligned}
\tag{8}
$$

an identity of degree four.

It is easily verified that

$$
\begin{aligned}
&\vartheta_1'^2\phi_{pqr}(a, b)\\
&\quad=\phi_{pst}(a+x, b-x)\phi_{sqm}(a, x)\phi_{trn}(b, -x)\vartheta_m(x)\vartheta_n(-x),
\end{aligned}
\tag{9}
$$

where each of the letters s, t, m, n is any one of the four theta subscripts.

This relation enables us not only to increase the degree, but also with each application, to increase the number of variables by one. Applying (9) to (4) we have the new identity

$$\phi_{111}\left(\frac{w+z+2x}{2},\ u-x\right)\phi_{111}\left(\frac{w+z}{2},\ x\right)$$

$$\cdot\,\phi_{111}(u,-x)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-u}{2},\ q^{1/2}\right)$$

$$+\,\phi_{111}\left(\frac{w-z+2x}{2},\ -u-x\right)\phi_{111}\left(\frac{w-z}{2},\ x\right)$$

$$\cdot\,\phi_{111}(-u,-x)\vartheta_3\left(\frac{v-z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+u}{2},\ q^{1/2}\right)$$

$$-\,\phi_{111}\left(\frac{w-z+2x}{2},\ v-x\right)\phi_{111}\left(\frac{w-z}{2},\ x\right)$$

$$\cdot\,\phi_{111}(v,-x)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w-v}{2},\ q^{1/2}\right)$$

$$-\,\phi_{111}\left(\frac{w+z+2x}{2},\ -v-x\right)\phi_{111}\left(\frac{w+z}{2},\ x\right)$$

$$\cdot\,\phi_{111}(-v,-x)\vartheta_3\left(\frac{u+z}{2},\ q^{1/2}\right)\vartheta_3\left(\frac{w+v}{2},\ q^{1/2}\right)\equiv 0.$$

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ON PRIMARY NORMAL DIVISION ALGEBRAS OF DEGREE EIGHT*

BY A. A. ALBERT

1. *Introduction.* A normal division algebra A of degree n over F will be called *primary* if A is not expressible as a direct product of two normal division algebras B and C, where neither B nor C has degree unity. It is well known that necessarily $n=p^e$, p a prime, if A is primary. Moreover, if $n=p^e$, then a *sufficient* condition that A be primary is that A shall have exponent† n.

I have recently proved‡ that if A has degree four then A is primary if and only if A has exponent four. In the present paper I shall prove that there exist primary (cyclic) normal division algebras of degree eight but exponent four so that the above sufficient condition is actually not necessary.§

2. *Cyclic Fields of Degree Eight.*‖ Let F be any non-modular field, and let C be a cyclic field of degree eight over F. Then I have proved that $C=F(x)$ contains a sub-field $F(y)$ which is cyclic of degree four over F and is defined by

$$y^2 = \nu(u - \tau),\ u^2 = \tau = 1 + \epsilon^2, \tag{1}$$

where $\nu\neq 0$, $\epsilon\neq 0$ are in F and $\tau=1+\epsilon^2$ is not the square of any quantity of F. I have also proved that

$$-\nu\tau = \xi_1^2 + \xi_2^2\tau,\ -\epsilon = (\eta_1^2 - \eta_2^2\tau)(\xi_1^2 + \xi_2^2\tau), \tag{2}$$

for ξ_1, ξ_2, η_1, η_2 in F. Conversely I have shown that if (1) and (2) are satisfied, then there exists a uniquely defined cyclic field $C=F(x)$ of degree eight over F and with $F(y)$ as cyclic quartic sub-field.

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* Presented to the Society, February 25, 1933.

† The exponent ρ of A of degree n is the least integer such that the direct power A^ρ is a total matric algebra, and is a divisor of n.

‡ Transactions of this Society, vol. 34 (1932), pp. 363–372.

§ It seems likely, however, that A of degree p^2, p a prime, is primary if and only if A has exponent p^2.

‖ For proofs of the results of this section see my paper on cyclic fields of degree eight which has been offered for publication to the editors of the Transactions of this Society.

In particular let F contain no quantity i such that $i^2=-1$, and let also $-\tau$ be not the square of any quantity of F. Then I have proved that the solution of $(2)_2$ is equivalent to the solution of

$$-\epsilon = \lambda^2 - \mu^2\tau, \tag{3}$$

for $\lambda=\lambda_1+\lambda_2 i$, $\mu=\mu_1+\mu_2 i$, $i^2=-1$, and $\lambda_1, \lambda_2, \mu_1, \mu_2$ in F, such that

$$\lambda_1\lambda_2 = \mu_1\mu_2\tau, \qquad \lambda_1\eta_2 = \mu_1\eta_1, \qquad \lambda_2\eta_1 = \mu_2\eta_2\tau, \tag{4}$$

while ξ_1 and ξ_2 are determined by

$$\xi_1 = \frac{\mu_1\eta_1 - \lambda_1\eta_2\tau}{\eta_1^2 - \eta_2^2\tau}, \quad \xi_2 = \frac{\mu_2\eta_1 - \lambda_2\eta_2}{\eta_1^2 - \eta_2^2\tau}, \quad \eta_1^2 - \eta_2^2\tau \neq 0. \tag{5}$$

We may evidently take $\lambda=\lambda_2 i$, $\mu=0$, $\epsilon=-\lambda^2=\lambda_2^2$ and (3) is satisfied. We also have $\lambda_1\eta_2=\mu_1\eta_1=\lambda_1\lambda_2=\mu_1\mu_2\tau=0$, while $\lambda_2\eta_1=\mu_2\eta_2\tau=0$ implies that $\eta_1=0$, and $\eta_2\neq 0$ is arbitrary. Hence, from (5), $\xi_1=0$, while $\xi_2=-\lambda_2(\eta_2\tau)^{-1}$. We have therefore proved that in this case

$$-\nu = \xi_2^2 = \left[\epsilon(\eta_2\tau)^{-1}\right]^2,$$

so that, for properly chosen η_2, we have $\nu=-1$.

THEOREM 1. *Every t in F such that*

$$\epsilon = t^2, \ \tau = 1 + \epsilon^2, \tag{6}$$

and $\pm\tau$ is not the square of any quantity of F, defines a cyclic field $F(x)$ of degree eight over F with a cyclic quartic sub-field

$$y^2 = \tau - u, \ u^2 = \tau, \tag{7}$$

so that $\nu=-1$.

3. *On a Rational Quadratic Form.* Let

$$ax^2 + by^2 + cz^2 + dw^2 \tag{8}$$

be an indefinite form with integer a, b, c, d. Then we may define (i, j) to be the (positive) greatest common divisor of any two integers i and j and may write

$$\begin{cases} a = (a,b)\cdot(a,c)\cdot(a,d)\cdot a_0, & b = (b,a)\cdot(b,c)\cdot(b,d)\cdot b_0, \\ c = (c,a)\cdot(c,b)\cdot(c,d)\cdot c_0, & d = (d,a)\cdot(d,b)\cdot(d,c)\cdot d_0. \end{cases} \tag{9}$$

If no three of a, b, c, d have a common factor greater than unity, it is well known* that (8) is a zero form only if

$$(10) \qquad -(a, b)(a, d)(b, c)(c, d)a_0c_0$$

is a quadratic residue of every odd prime p dividing either (a, c) or (b, d) for which $a_0b_0c_0d_0$ is a quadratic residue of p.

We shall take the t of Theorem 1 to be even so that ϵ is even and τ is odd. Consider the form

$$(11) \quad \tau x^2 - (4\tau^3 - 1)y^2 - 2\tau(2\tau\epsilon + 1)[z^2 - (4\tau^3 - 1)w^2],$$

which is a zero form if and only if

$$(12) \quad x^2 - \tau(4\tau^3 - 1)y^2 - 2(2\tau\epsilon + 1)[z^2 - (4\tau^3 - 1)w^2]$$

is a zero form. In (12) we write

$$(13) \qquad \begin{gathered} a = 1,\ b = -\tau(4\tau^3 - 1),\ c = -2(2\tau\epsilon + 1), \\ d = 2(2\tau\epsilon + 1)(4\tau^3 - 1); \end{gathered}$$

but $4\tau^3-1=4\tau^2(1+\epsilon^2)-1=(2\tau\epsilon+1)(2\tau\epsilon-1)+4\tau^2$ is evidently prime to $2(2\tau\epsilon+1)\tau$. Hence no three of a, b, c, d have a factor in common. In fact

$$(14) \qquad \begin{cases} a = a_0 = (a, b) = (a, c) = (a, d) = (b, c) = 1, \\ (b, d) = (4\tau^3 - 1), \\ (c, d) = 2(2\tau\epsilon + 1),\ b_0 = -\tau,\ c_0 = -1,\ d_0 = 1, \end{cases}$$

so that, from (10),

$$(15) \quad -(a, b)(a, d)(b, c)(c, d)a_0c_0 = (c, d) = 2(2\tau\epsilon + 1)$$

is a quadratic residue of every odd prime p dividing $(b, d)=h$ $=4\tau^3-1$ such that $a_0b_0c_0d_0=\tau$ is a quadratic residue of p. But if $4\tau^3-1\equiv 0 \pmod{p}$, then for the Legendre symbol $(\tau \mid p)$,

$$(16) \qquad (\tau \mid p) = (4\tau^3 \mid p) = (1 \mid p) = 1.$$

Hence every prime p dividing h has the above property. It follows that if we write $2g=2(2\tau\epsilon+1)$, then $2g$ must be a quadratic residue of h. Hence the Jacobi symbol $(2g \mid h)=1$. But $(h \mid g)=1$, since $h=(2\tau\epsilon-1)g+4\tau^2\equiv 4\tau^2 \pmod{g}$. Then,†

* See L. E. Dickson, *Studies in the Theory of Numbers*, p. 71.

† We use, of course, the laws $(g \mid h)\cdot(h \mid g)=(-1)^{(g-1)/2\cdot(h-1)/2}$, $(2 \mid h)=(-1)^{(h^2-1)/8}$, for computing $(2g \mid h)$.

$$(17) \qquad (2g \mid h) = (2 \mid h)(g \mid h) = (-1)^{\alpha},$$

where

$$(18) \quad \begin{cases} \alpha = \dfrac{h^2-1}{8} + \dfrac{h-1}{2}\cdot\dfrac{g-1}{2} = \dfrac{h-1}{8}[h+1+2g-2] \\ \quad = \dfrac{4\tau^3-2}{2}\cdot\dfrac{4\tau^3+4\tau\epsilon}{4} = (2\tau^3-1)\tau(\tau^2+\epsilon). \end{cases}$$

But $2\tau^3-1$, τ, τ^2 are odd while $\epsilon = t^2$ is even. Hence α is odd, so that $(2g \mid h) = -1$, a contradiction.

For every even integer t the integer $1+t^4=\tau$ is positive and hence $-\tau \neq x^2$ for rational x. Also $t=2\xi$, $\tau = 1+16\xi^4$ is not identically the square of any polynomial in ξ. It follows that the equation $x^2=\tau$ is irreducible in $R(\xi)$ and, by the Hilbert Irreducibility Theorem, there exist infinitely many integer values of ξ for which $\tau \neq x^2$ for any rational x.

THEOREM 2. *There exist infinitely many cyclic fields of degree eight over R for which the quadratic form* (11) *is not a zero form.*

4. *On Function Fields.* In this section we shall obtain some quite simple theorems to be used later. Let F be any nonmodular field and $y, z, \cdots, w$ be independent indeterminates. Then $F(y, \cdots, w)$ is defined to be the *field* of all rational functions with coefficients in F of $y, \cdots, w$. Also $F[y, \cdots, w]$ is defined to be the *domain of integrity* of all polynomials with coefficients in F of $y, \cdots, w$. Suppose that f is in $F[y]$ and $f = f_1 \cdot f_2$, where the f_i are in $F[y, z]$. The degree of f in z is the sum of the degrees of f_1 and f_2 in z and is zero. Hence we have the following lemma.

LEMMA 1. *If f is in $J = F[y]$, and is a product of two quantities of $F[y, z]$, then these quantities are in J.*

Let next f be in J and let f_1 and f_2 be in $K[y]$, where $K = F(z)$. Then we may write

$$f_1 = \frac{g_1}{h_1}, \quad f_2 = \frac{g_2}{h_2},$$

where g_i is a polynomial in y with coefficients in $F[z]$, h_i is in $F[z]$ and has no factor in $F[z]$ which divides all the coefficients of g_i. But if $f = f_1 \cdot f_2$, then

$$f = \frac{g_1 g_2}{h_1 h_2}$$

is a *polynomial* in y with coefficients in F so that h_1 divides g_2 and h_2 divides g_1. Hence $g_1 = h_2 k_1$, $g_2 = h_1 k_2$, so that

$$f = f_1 f_2 = (k_1 h^{-1})(k_2 h), \qquad h = h_1/h_2.$$

By Lemma 1, k_1 and k_2 have coefficients in F.

LEMMA 2. *Let f be a polynomial in y with coefficients in F. Then, if y and z are indeterminates, f is reducible in $F(z)$ if and only if f is reducible in F.*

By induction the above result may be immediately extended to the case of an arbitrary number of variables y_i and indeterminates z_i. Also, an application of Lemma 2 to the Galois resolvent of any equation without multiple roots yields the following theorems.

THEOREM 3. *Let $f(x) = 0$ have coefficients in F and group G for F. Then, if z is an indeterminate, the group of $f(x) = 0$ for $F(z)$ is also G.*

THEOREM 4. *Let $W = F(x)$ be an algebraic field over F with G as its group of automorphisms. Then $W' = F(x, z)$ is an algebraic field over $F(z)$, with the same degree and group G as W, for any indeterminate z.*

5. *On a Quadratic Form over $R(z)$.* We let R be the field of all rational numbers, z be a parameter, $K = R(z)$, $J = R[z]$. Let $R(x)$ be a cyclic field of degree eight over R chosen so that (11) is not a zero form. By Theorem 4 the field $K(x)$ is cyclic of degree eight over K. Consider the form

$$(19)\quad P(\lambda_1, \cdots, \lambda_6) \equiv \lambda_1^2 \tau + (\lambda_2^2 - \lambda_3^2 \tau)\beta - \lambda_4^2 \gamma - (\lambda_5^2 - \lambda_6^2 \gamma)\delta,$$

in the variables $\lambda_1, \cdots, \lambda_6$ in K, where, since $\nu = -1$,

$$(20)\quad \begin{aligned} \gamma &= \beta_2^2 \tau - \beta_1^2, \ \beta = 2\beta_1 \gamma, \\ \delta &= 2\nu\tau(\beta_2 \epsilon - \beta_1)\beta_1 = -2\tau\beta_1(\beta_2 \epsilon - \beta_1). \end{aligned}$$

Write $\beta_1 = -z$, $\beta_2 = 2\tau z$ and, since β, δ, γ have the factor z^2, the form $P(\lambda_1, \cdots, \lambda_6)$ becomes

$$(21)\quad \begin{cases} Q(\mu_1, \cdots, \mu_6) \equiv \mu_1^2\tau - (4\tau^3 - 1)\mu_2^2 \\ \qquad\qquad - 2\tau(2\tau\epsilon + 1)[\mu_3^2 - (4\tau^3 - 1)\mu_4^2] \\ \qquad\qquad - 2_2(4\tau^3 - 1)(\mu_5^2\tau - \mu_6^2), \end{cases}$$

where $\mu_1 = \lambda_1$, $\mu_2 = \lambda_4 z$, $\mu_3 = \lambda_5 z$, $\mu_4 = \lambda_6 z^2$, $\mu_5 = \lambda_2 z$, $\mu_6 = \lambda_3 z$. Evidently (19) is a zero form if and only if (21) is a zero form. Also (21) vanishes for values not all zero of the μ_i in K if and only if it vanishes for μ_i not all zero in J.

Let us write (21) in the form $Q \equiv 0$ in z, that is,

$$(22)\quad \mu_1^2\tau - (4\tau^3 - 1)\mu_2^2 - 2\tau(2\tau\epsilon + 1)[\mu_3^2 - (4\tau^3 - 1)\mu_4^2] \equiv 2z(4\tau^3 - 1)(\mu_5^2\tau - \mu_6^2),$$

and designate the left member by S. Every term $\alpha_i\mu_i^2$ of S has rational α_i and *even* degree in z. Hence S has *odd* degree if and only if it has leading coefficient

$$x_1^2\tau - (4\tau^3 - 1)x_2^2 - 2\tau(2\tau\epsilon + 1)[x_3^2 - (4\tau^3 - 1)x_4^2]$$

formally equal to zero for $x_1, \cdots, x_4$ not all zero in R. This is contrary to our choice of $R(x)$. Hence S has even degree, $S \equiv 0$ only if $\mu_1 = \cdots = \mu_4 \equiv 0$. But $S \equiv T \equiv 2z(4\tau^3 - 1)(\mu_5^2\tau - \mu_6^2)$, so that T has even degree. Hence $\mu_5^2\tau - \mu_6^2$ has odd degree and thus has *formal* leading coefficient zero. But this coefficient is $x_5^2\tau - x_6^2$, which must be zero for x_5 and x_6 in R and not both zero unless $T \equiv 0$. But this is impossible by our choice of τ. Hence $T \equiv 0$, $S \equiv 0$, a contradiction. For we have proved now that S and T are both zero only when all the μ_i are zero.

THEOREM 5. *There exist cyclic fields of degree eight over K for which* (19) *does not vanish for $\lambda_1, \cdots, \lambda_6$ in K and not all zero.*

6. *A Type of Normal Division Algebra.* Let F be any non-modular field. It is well known that every normal simple algebra of degree two over F is an algebra

$$(23)\qquad Q(\alpha, \beta) = (1, i, j, ij),$$

where $\alpha \neq 0$, $\beta \neq 0$ are in F. Let $Z = F(d)$ be a cyclic field of degree four over F,

$$(24)\qquad d^2 = \nu(u - \tau),\ u^2 = \tau = 1 + \epsilon^2,$$

where $\nu \neq 0$ is in F, ϵ in F. Define

$$D = (Z, S, \gamma) \tag{25}$$

to be the cyclic algebra

$$(d^i e^j),\ e^j a = a^{S^j} e^j, \quad (i, j = 0, 1, 2, 3), \tag{26}$$

where if $a = a(d)$ is in Z, then

$$\begin{gathered} a^S \equiv a[\theta^i(d)],\ \theta(d) = d\left(\frac{u+1}{\epsilon}\right), \\ \theta^2(x) = -x,\ \theta^3(x) = -\theta(x),\ \theta^4(x) = x. \end{gathered} \tag{27}$$

I have proved that if D has exponent $\rho < 4$, then

$$\gamma = \beta_2^2 \tau - \beta_1^2,$$

for β_1 and β_2 in F, and conversely. I have also shown* that when $\rho < 4$

$$D = Q(\alpha, \beta) \times Q(\gamma, \delta) \tag{28}$$

is a direct product of two generalized quaternion algebras, where

$$\alpha = \tau,\ \beta = 2\beta_1\gamma,\ \gamma = \beta_2^2\tau - \beta_1^2,\ \delta = 2\nu\tau\beta_1(\beta_2\epsilon - \beta_1). \tag{29}$$

I have also proved that if (28) is satisfied, then D is a division algebra if and only if (19) is not a zero form. Hence, if $F = R(z)$, we have already proved that D is a division algebra of exponent two. We therefore have the following theorem.

THEOREM 6. *There exist cyclic fields of degree eight over $K = R(z)$ with corresponding sub-fields Z of degree four over K such that K contains quantities γ in F for which $D = (Z, S, \gamma)$ is a division algebra of exponent two.*

The above theorem provides the first proof† of the existence of a non-primary normal division algebra of degree four over a function field $R(z)$, the simplest previous example being that of an algebra over $R(y, z)$, where y and z are two independent indeterminates.

7. *Primary Algebras of Degree Eight.* Let A be a normal division algebra of degree p^e over F, p a prime. I have proved‡ that

$$A^p = M_{\theta}^{p-1} \times M_p \times A_p = M \times A_p,$$

* This Bulletin, vol. 37 (1931), pp. 301–312, Theorems 3, 4.

† It is not difficult, however, to give a simpler proof for the case $\nu > 0$.

‡ See Theorem 7 of my paper *Algebras of degree* 2^a, etc., Annals of Mathematics, vol. 33 (1932), pp. 311–318.

where $M = M_0^{p^{e-1}} \times M_p$, M_0 is a total matric algebra of degree p^e, M_p is a total matric algebra of degree $p^f > 1$, A_p is a normal division algebra of degree $p^{e-f} \geqq 1$. Also the exponent ρ of A has the value $\rho = p^r$, where $r \leqq e$. Since $\rho = p\rho_1$, we have $\rho_1 = p^{r-1}$, $A^p = A^{p\rho_1} = M^{\rho_1} \times (A_p)^{\rho_1}$ is a total matric algebra. Hence $A_p{}^{\rho_1}$ is a total matric algebra. Conversely, if ρ_0 is the exponent of A_p, then $A^{p\rho_0} = M^{\rho_0} \times A_p{}^{\rho_0}$ is a total matric algebra. We thus have $\rho \leqq p\rho_0$, $p\rho_1 \leqq p\rho_0$, $\rho = p\rho_1 = p\rho_0$, and we have the following fact.

LEMMA 3. *The exponent ρ of A is $p\rho_0$, where ρ_0 is the exponent of A_p.*

Let $p = 2$, $e = 3$, so that $A^2 = (M_0 \times M_2) \times A_2$. If $A = B \times C$ is non-primary, where B may be taken to have degree four, C degree two, then $A^2 = B^2 \times C^2 = (L \times L_2 \times B_2) \times (K \times K_2 \times C_2)$. But K_2 has degree two, C_2 has degree unity, C^2 is a total matric algebra. Hence $A^2 = M_0 \times Q$, where M_0 is a total matric algebra and Q is a normal simple algebra of degree two.

LEMMA 4. *Let A be a non-primary normal division algebra of degree eight. Then $A^2 = M_0 \times Q$, where M_0 is a total matric algebra and Q is a normal simple algebra of degree two.*

We now let A be any normal division algebra of degree eight. Then $A^2 = M \times A_2$, where M is a total matric algebra and A_2 is a normal division algebra. If A_2 has degree unity or two, then A has exponent two or four by Lemma 3 and may or may not be primary. But, by Lemma 3, if A_2 has degree four then A is primary and has exponent $2s$ where s is the exponent of A_2.

THEOREM 7. *Let A be a normal division algebra of degree eight over F so that*

$$A^2 = M \times A_2,$$

where M is a total matric algebra and A_2 is a normal division algebra of degree $2^t \leqq 4$. Then A has exponent two or four according as $t = 0, 1$ and it is not yet known whether A may be primary. But if $t = 2$, then A is primary and has exponent $2s$, where s is the exponent of A_2.

By Theorems 6, 7 we have the following result.

THEOREM 8. *There exist primary normal division algebras of degree eight and exponent four.*

THE UNIVERSITY OF CHICAGO

CONVEX DOMAINS AND LINEAR COMBINATIONS OF CONTINUOUS FUNCTIONS*

BY I. J. SCHOENBERG

1. *Introduction. Dines' Problem.* Let there be given n real, continuous and linearly independent functions

$$\phi_1(x), \phi_2(x), \cdots, \phi_n(x), \qquad (a \leqq x \leqq b), \tag{1}$$

which we may interpret as cartesian coordinates of an arc of curve Γ in the n-space S_n. Let $K = K(\Gamma)$ denote the smallest closed convex domain in S_n containing the arc Γ. The following lemma is due to F. Riesz.†

LEMMA 1. *A point $X = (x_1, x_2, \cdots, x_n)$ belongs to K if and only if its coordinates can be represented in the form*

$$x_k = \int_a^b \phi_k(x)d\psi(x), \qquad (k = 1, 2, \cdots, n), \tag{2}$$

where $\psi(x)$ is monotonic and non-decreasing in $[a, b]$, with $\psi(a) = 0$, $\psi(b) = 1$.

A characterization of the points which are *interior* to K is given by the following lemma.

LEMMA 2. *A point $X = (x)$ is an interior point of K if and only if its coordinates can be represented in the form*

$$x_k = \int_a^b p(x)\phi_k(x)dx, \quad (k = 1, 2, \cdots, n), \tag{3}$$

where $p(x)$ is continuous‡ and positive in $[a, b]$ and $\int_a^b p(x)dx = 1$.

. This lemma is proved in §4 below. I wish to show now how by

* Presented to the Society, April 14, 1933.

† F. Riesz, Annales de l'École Normale, (3), vol. 28, pp. 56–57. It can be derived immediately from some fundamental properties of convex domains to be found in Chapter I of C. Carathéodory's article *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen*, Rendiconti di Palermo, vol. 32 (1911), pp. 193–217. The reader is referred to this place for all the information about convex domains which is used in this note.

‡ Concerning this assumption on $p(x)$ see the concluding remark at the end of this note.

means of these lemmas some interesting questions about linear combinations of the functions (1) can be readily answered. We start with the following problem stated and solved by L. L. Dines.*

Under what conditions does every real linear combination

$$(4)\quad \Phi(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x), \quad \left(\sum_1^n a_k^2 > 0 \right),$$

change sign in the interval $[a, b]$?

That this should be so, it is obviously necessary and sufficient that every hyperplane

$$(5)\quad \pi(x) \equiv a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0, \qquad \left(\sum_1^n a_k^2 > 0 \right),$$

passing through the origin O of S_n shall cut† the arc Γ, that is, that the origin $O=(0, 0, \cdots, 0)$ be an *interior* point of K. From this remark and Lemma 2, we immediately derive the following answer.

THEOREM 1 (L. L. DINES). *Every linear combination* (4) *will change sign in* $[a, b]$ *if and only if there exists a positive continuous function* $p(x)$ *with*

$$\int_a^b p(x)\phi_k(x)dx = 0, \quad (k = 1, 2, \cdots, n).$$

By similar reasons every $\Phi(x)$ will vanish somewhere in $[a, b]$ if and only if the origin O is a point of K. Hence Lemma 1 gives the following result.

THEOREM 2. *Every linear combination* (4) *will vanish somewhere in* $[a, b]$ *if and only if there exists a monotonic function* $\psi(x)$ *not identically constant in* $[a, b]$ *with*

$$\int_a^b \phi_k(x)d\psi_k(x) = 0, \quad (k = 1, 2, \cdots, n).$$

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* L. L. Dines, *A theorem on orthogonal functions with an application to integral inequalities*, Transactions of this Society, vol. 30 (1928), pp. 425–438. See also L. L. Dines, Annals of Mathematics, (2), vol. 28 (1926), pp. 393–395, and N. H. McCoy, this Bulletin, vol. 36 (1930), pp. 878–882.

† That is to say, have points of the arc Γ on both of its sides.

2. *Generalization of Dines' Problem.* A recent article of J. Favard* suggests the following generalization of the problem of Dines: *For what constants* $c_1, c_2, \cdots, c_n$, $(\Sigma c_k^2 > 0)$, *does every linear combination* (4), *whose coefficients satisfy the relation*

$$a_1c_1 + a_2c_2 + \cdots + a_nc_n = 0, \tag{6}$$

change sign in $[a, b]$?

An answer to this question is given in the following theorem.

THEOREM 3. *Every linear combination* (4), *whose coefficients satisfy the relation* (6), *changes sign in* $[a, b]$ *if and only if the system of equations*

$$\int_a^b p(x)\phi_k(x)dx = c_k, \quad (k = 1, 2, \cdots, n), \tag{7}$$

has a continuous solution $p(x) \neq 0$ *in* $[a, b]$.†

The proof of this theorem requires the following additional geometric lemma whose proof we postpone to the end of this note (§5).

LEMMA 3. (a) *If a straight line* Δ *of* S_n *has no point in common with* K, *then we can pass through* Δ *a hyperplane* π *which also has no point in common with* K, *that is,* π *is a bound of* K. (b) *If* Δ *and* K *have points in common none of which is interior to* K (*we may call* Δ *a line of support of* K), *then we can pass through* Δ *a hyperplane of support* π *of* K.

Let us return to the proof of Theorem 3. Let Δ denote the straight line joining the point $C = (c_1, c_2, \cdots, c_n)$ to the origin O. Lemma 3 gives the following geometric criterion: *Every* $\Phi(x)$, *satisfying* (6), *will change sign in* $[a, b]$ *if and only if* Δ *contains an interior point* P *of* K. Indeed, if Δ contains such a point P, then every hyperplane (5) through Δ will cut the arc Γ, i.e., every $\Phi(x)$ satisfying (6) will change sign in $[a, b]$. Conversely, if Δ contains no point interior to K, then Lemma 3 shows that there exists a linear combination $\Phi(x)$, satisfying (6), which

* J. Favard, *Sur les zéros réels des polynômes*, Bulletin de la Société Mathématique, vol. 59 (1931), pp. 229–255.

† If every $\Phi(x)$ changes sign in $[a, b]$, irrespective of (6), then system (7) has a required solution $p(x)$ for any constants $c_1, c_2, \cdots, c_n$. Indeed, the origin O is then interior to K.

does not change sign in $[a, b]$. Moreover, if Δ contains interior points of K, it will also contain such a point $P \neq O$. The coordinates of P expressed in the form (3) and multiplied by the ratio $\overline{OC}/\overline{OP}(\neq 0)$, yield the characteristic representation (7) of the coordinates of C. Similarly, Theorem 2 is generalized by the following theorem.

THEOREM 4. *We assume that not every linear combination* (4) *vanishes somewhere in* $[a, b]$. *Under this assumption every* $\Phi(x)$, *whose coefficients satisfy* (6), *will vanish somewhere in* $[a, b]$ *if and only if the system*

$$\int_a^b \phi_k(x)d\psi(x) = c_k, \qquad (k = 1, 2, \cdots, n), \tag{8}$$

has a monotonic solution $\psi(x)$ *not identically constant in* $[a, b]$.

Indeed, the first assumption of this theorem means that O is an exterior point of K. Then relation (6) implies that $\Phi(x)$ vanishes in $[a, b]$ if and only if the straight line Δ passing through O and C has at least one point P in common with K (Lemma 3, case (a)). The coordinates of P expressed in the form (2), and multiplied by the ratio $\overline{OC}/\overline{OP}(\neq 0)$, yield the characteristic representation (8) of the coordinates of C. It is easy to see geometrically that Theorem 4 fails to be always true if we drop the assumption of its first sentence.

3. *Infinite Sequences of Functions.* Let there be given an infinite sequence of linearly independent continuous functions

$$\phi_0(x) \equiv 1, \ \phi_1(x), \ \phi_2(x), \ \cdots, \qquad (a \leqq x \leqq b), \tag{9}$$

and let

$$c_0 > 0, c_1, c_2, \cdots, \tag{10}$$

be a given sequence of real constants. Again it is suggested by Favard's paper that we ask the following question. *For what sequences* (10) *does, for every n, a linear combination*

$$\Phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x), \tag{11}$$

whose coefficients satisfy the relation

$$a_0c_0 + a_1c_1 + \cdots + a_nc_n = 0, \tag{12}$$

always vanish, at least once, in $[a, b]$?

We shall prove the following theorem.

THEOREM 5. *For every n, a linear combination* (11), *whose coefficients satisfy* (12), *will always vanish at least once in* $[a, b]$ *if and only if the system*

$$\int_a^b \phi_k(x)d\psi(x) = c_k, \quad (k = 0, 1, 2, \cdots) \tag{13}$$

admits a monotonic solution $\psi(x)$.

The condition of this theorem is obviously sufficient. Its necessity is readily derived from Theorem 4 and a well known theorem of Helly.*

Indeed, on account of Theorem 4, the finite system

$$\int_a^b \phi_k(x)d\psi_n(x) = c_k, \ (k = 0, 1, 2, \cdots, n),$$

admits, for every n, a monotonic solution $\psi_n(x)$, with $\psi_n(a) = 0$. By Helly's theorem there exists a subsequence $\psi_{n_i}(x)$ which converges to a monotonic solution $\psi(x)$ of the system (13).

For the special sequence of functions

$$\phi_0(x) = 1, \ \phi_1(x) = x, \ \phi_2(x) = x^2, \ \cdots, \tag{14}$$

Theorem 5 was given by Favard (loc. cit. p. 244). For this particular sequence (14), Favard proved Theorem 5 also when one or both of the limits a and b are infinite. Our method can not cover these cases, since Theorem 4, on which it is based, fails to be true if $a = -\infty$, $b = +\infty$, $n \geqq 3$, for example, if $c_0 = 1$, $c_1 = c_2 = \cdots = c_{n-1} = 0$, $c_n = 1$.†

4. *Proof of Lemma* 2. The point $X = (x)$ given by (3) is interior to K. Indeed, assuming for the moment that X is on the boundary of K, there is no loss of generality (suitable change

* E. Helly, Sitzungsberichte der Wiener Akademie, vol. 121 IIa (1912), p. 286 and pp. 288–9.

† Indeed, the polynomial $\Phi(x) = a_0 + a_1x + \cdots + a_nx^n$ necessarily has a real root if $a_0 + a_n = 0$, while the corresponding system

$$\int_{-\infty}^{\infty} d\psi = 1, \ \int_{-\infty}^{\infty} xd\psi = \int_{-\infty}^{\infty} x^2d\psi = \cdots = \int_{-\infty}^{\infty} x^{n-1}d\psi = 0, \ \int_{-\infty}^{\infty} x^n d\psi = 1$$

has no monotonic solution $\psi(x)$.

of axes) if we assume that the hyperplane $x_1=0$ is a plane of support of K, that is

$$\phi_1(x) \geqq 0 \text{ for } a \leqq x \leqq b \text{ and } \int_a^b p(x)\phi_1(x)dx = 0.$$

These last conditions, however, are in contradiction with $p(x)>0$ and $\phi_1(x)\neq 0$ in $[a, b]$. The sufficiency of the condition of Lemma 2 is thus established.

Let $a=0$, $b=1$, and let K_0 be the totality of points $X=(x)$ representable in the form (3) with $p(x)$ continuous and positive in $[0, 1]$ and $\int_0^1 p(x)dx=1$. The point set K_0 is obviously convex. We shall prove now that *every point* $P=(p_1, p_2, \cdots, p_n)$ *of* K *is a limit point of the point set* K_0. Indeed, by Lemma 1 we may write

$$(15)\quad p_k = \int_0^1 \phi_k(x)d\psi(x),\ \psi(x) \text{ monotonic},\ \psi(0)=0,\ \psi(1)=1,$$

for $k=1, 2, \cdots, n$. Consider now the polynomial of degree $m(>1)$

$$(16)\quad P_m(x) = \frac{m}{m+2}\left\{B_m(\psi, x) + \frac{1}{m}x^m - \frac{1}{m}(1-x)^m + \frac{1}{m}\right\},$$

where

$$(17)\qquad B_m(\psi, x) = \sum_{\mu=0}^{m}\psi\left(\frac{\mu}{m}\right)\binom{m}{\mu}x^\mu(1-x)^{m-\mu}$$

is the mth Bernstein polynomial of $\psi(x)$.* From

$$P_m'(x) = \frac{m}{m+2}\left\{m\sum_{\mu=0}^{m-1}\left[\psi\left(\frac{\mu+1}{m}\right) - \psi\left(\frac{\mu}{m}\right)\right]\binom{m-1}{\mu}x^\mu(1-x)^{m-\mu-1} + x^{m-1} + (1-x)^{m-1}\right\},$$

we see that

$$(18)\qquad P_m'(x) > 0,$$

* S. Bernstein, Communications Charkow Mathematical Society, (2), vol. 13 (1912), pp. 1–2.

for $0 \leqq x \leqq 1$. Moreover, (16) and (17) give

$$P_m(0) = 0, \quad P_m(1) = 1. \tag{19}$$

Finally, a fundamental convergence property of the polynomials of Bernstein shows that

$$\lim_{m \to \infty} P_m(x) = \psi(x) \tag{20}$$

for every value of x where $\psi(x)$ is continuous. The point

$$X^{(m)}: \qquad x_k^{(m)} = \int_0^1 \phi_k(x) P_m'(x) dx, \quad (k = 1, 2, \cdots, n),$$

on account of (18) and (19), belongs to the point set K_0; and, by (20) and Helly's theorem, we have $X^{(m)} \to P$ as $m \to \infty$.

If Q is a fixed *interior* point of K, then Q is also *interior* to a suitable polyhedron with $n+1$ vertices $P_0, P_1, \cdots, P_n$, all of which belong to K. By our last result we can find $n+1$ points $X_0, X_1, \cdots, X_n$ of K_0 such that $X_i (i = 0, 1, \cdots, n)$, is arbitrarily close to P_i. For such a sufficiently close approximation the point Q will clearly be a point of the polyhedron $X_0 X_1 \cdots X_n$, hence, a point of K_0, since K_0 is convex. This concludes the proof of Lemma 2.

5. *Proof of Lemma 3.* (a) Let Δ have no common point with K. Let $\overline{AB}$ be the shortest distance from Δ to K, the point A belonging to Δ and B to K. Then Δ is perpendicular to the segment AB so that the plane π, perpendicular to AB at A, will contain the straight line Δ. This plane π has no point in common with K (see Carathéodory, loc. cit. p. 198).

(b) Let Δ be a line of support of K. Let I be a fixed interior point of K and let P be a common point of Δ and K. Join I and P by a line segment and take a point P' on its extension beyond the point P. Draw through P' the line Δ' parallel to Δ. *Then Δ' and K have no common point.* For if they had a common point Q', then the point of intersection Q of Δ and IQ' would be an interior point of K, contradicting the fact that Δ is a line of support of K. According to (a) we can pass through Δ' a bound π' of K. If now P' approaches P by moving on the line $P'PI$, the corresponding plane π' will move parallel to Δ. For a countable number of suitably chosen positions of P', the correspond-

ing plane π' will converge* to a limiting plane π, which necessarily contains Δ, and which, as a limit of bounds π' of K, is a plane of support of K.

6. *Conclusion.* Finally, I wish to point out two facts. 1. The assumptions that the functions (1) and (9), respectively, are linearly independent are not essential for Theorems 2, 4, and for Theorem 5, respectively. 2. In Lemma 2, the continuity of $p(x)$ can be replaced by the stronger condition that $p(x)$ be a polynomial. To show this one has to change in the proof given in §4 the definition of K_0 which now should be the set of points given by (3), where $p(x)$ is a polynomial, positive in $[a, b]$, with $\int p dx = 1$. Our Theorems 1 and 3 can be changed accordingly, $p(x)$ being now a polynomial.†

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* See Carathéodory's proof (loc. cit. p. 198) for the existence of a plane of support of K passing through a given boundary point of K.

† After this note was in type, I noticed several articles that are closely connected with it. In two articles by S. Kakeya, *On some integral equations*, I and II (Tôhoku Mathematical Journal, vol. 4 (1914), pp. 186–190, and Proceedings of the Tôkyô Mathematical-Physical Society, (2), vol. 8 (1915–16), pp. 83–102), the possibility of finding a solution $p(x)$ of the system (7) is discussed thoroughly, the solution $p(x)$ being subject to various more or less complicated auxiliary conditions. The conditions $p(x) > 0$, $\int p dx = 1$, which lead to my Lemma 2 above, were not considered by Kakeya. A note by M. Fujiwara, *On the system of linear inequalities and linear integral inequality* (Proceedings of the Imperial Academy of Japan, vol. 4 (1928), pp. 330–333) seems to contain (p. 332) a proof of Dines' Theorem 1, similar to the proof given in the present note. It does not actually cover the theorem, however, since Fujiwara derives there not the condition $p(x) > 0$ of Theorem 1, but only the weaker (necessary but not sufficient) condition $p(x) \geqq 0$. Finally, I may mention a beautiful note by W. Fenchel, *Geschlossene Raumkurven mit vorgeschriebenem Tangentenbild* (Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 39 (1930), pp. 183–185), where Lemma 2 is proved by a simple elementary method which requires no limiting process whatever, except, of course, ordinary integration.

LINEAR INTEGRAL EQUATIONS OF FUNCTIONS OF TWO VARIABLES*

BY T. S. PETERSON

1. *Introduction.* It is the purpose of this paper to consider certain conditions for the solution of the following linear integral equation:

$$\bar{y}(\alpha,\beta) = y(\alpha,\beta) + \lambda\int_a^b K(\alpha,\sigma)y(\sigma,\beta)d\sigma + \mu\int_a^b L(\beta,\tau)y(\alpha,\tau)d\tau + \nu\int_a^b\int_a^b M(\alpha,\beta,\sigma,\tau)y(\sigma,\tau)d\sigma d\tau$$

and, particularly, the truncated form with $M(\alpha,\beta,\sigma,\tau)\equiv 0$. The more important results of the paper are to be found summarized in Theorems 2 and 3.

Throughout the paper we shall consider all given functions as bounded and continuous, and in order to facilitate the work we shall adhere to the notation (1) *to represent the variables of functions as indices*, (2) *to signify by the repetition of an index in a term, once as a subscript and once as a superscript, an integration on that variable over the fundamental interval* (a, b).

2. *A Generalization of the Fredholm Equation.* Let us consider a special type of integral equation of a function of two variables which has as its origin the succession of two ordinary Fredholm equations, namely

$$\bar{y}^{\alpha\beta} = y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta} + \mu L_\tau^\beta y^{\alpha\tau} + \lambda\mu K_\sigma^\alpha L_\tau^\beta y^{\sigma\tau}. \tag{1}$$

In fact, (1) is given by the succession of equations

$$z^{\alpha\beta} = y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta}, \quad \bar{y}^{\alpha\beta} = z^{\alpha\beta} + \mu L_\tau^\beta z^{\alpha\tau}. \tag{2}$$

The equations (2) being ordinary Fredholm equations, it is evident at once that the equation (1) has the unique, continuous inverse

$$y^{\alpha\beta} = \bar{y}^{\alpha\beta} + \lambda k_\sigma^\alpha \bar{y}^{\sigma\beta} + \mu l_\tau^\beta \bar{y}^{\alpha\tau} + \lambda\mu k_\sigma^\alpha l_\tau^\beta \bar{y}^{\sigma\tau}, \tag{3}$$

providing that λ and μ are not characteristic values of their re-

* Presented to the Society, December 29, 1932.

spective kernels. In equation (3) the small lettered functions represent the respective inverses to the large lettered functions, that is,

$$k_\beta^\sigma = -\frac{D_\beta^\sigma[\lambda K_\tau^\sigma]}{D[\lambda K_\tau^\sigma]},$$

where $D_\beta^\sigma[\lambda K_\tau^\sigma]$ and $D[\lambda K_\tau^\sigma]$ are the first Fredholm minor and the Fredholm determinant respectively. If either parameter has a characteristic value, say $\lambda = \bar{\lambda}$, then it is evident again that the homogeneous equation (1), that is, $\bar{y}^{\alpha\beta} \equiv 0$, has a solution. Moreover these solutions will be of the form $\Phi_i^\alpha \phi^\beta$, where ϕ^β is an arbitrary function and the Φ_i^α, $(i = 1, 2, \cdots, n)$, are the fundamental solutions of the equation

$$0 = f^\alpha + \bar{\lambda} K_\sigma^\alpha f^\sigma.$$

Further properties of equation (1) may be written down at once following the classical theory of Fredholm.

A general equation in which we are more interested, however, is of the following form:

$$\bar{y}^{\alpha\beta} = y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta} + \mu L_\tau^\beta y^{\alpha\tau} + \nu M_{\sigma\tau}^{\alpha\beta} y^{\sigma\tau}. \tag{4}$$

To solve equation (4), let us consider the succession of transformations

$$z^{\alpha\beta} = y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta}, \quad \bar{z}^{\alpha\beta} = z^{\alpha\beta} + \mu L_\tau^\beta z^{\alpha\tau}, \quad \bar{y}^{\alpha\beta} = \bar{z}^{\alpha\beta} + N_{\sigma\tau}^{\alpha\beta} \bar{z}^{\sigma\tau}. \tag{5}$$

These on substitution yield

$$\begin{aligned} \bar{y}^{\alpha\beta} = {} & y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta} + \mu L_\tau^\beta y^{\alpha\tau} + \{ N_{\sigma\tau}^{\alpha\beta} + \lambda N_{\xi\tau}^{\alpha\beta} K_\sigma^\xi \\ & + \mu N_{\sigma\eta}^{\alpha\beta} L_\tau^\eta + \lambda\mu N_{\xi\eta}^{\alpha\beta} K_\sigma^\xi L_\tau^\eta + \lambda\mu K_\sigma^\alpha L_\tau^\beta \} y^{\sigma\tau}, \end{aligned} \tag{6}$$

which, subject to the conditions that λ, μ, and 1 are not characteristic values of the kernels of (5), has the unique, continuous inverse

$$\begin{aligned} y^{\alpha\beta} = {} & \bar{y}^{\alpha\beta} + \lambda k_\sigma^\alpha \bar{y}^{\sigma\beta} + \mu l_\tau^\beta \bar{y}^{\alpha\tau} + \{ n_{\sigma\tau}^{\alpha\beta} + \lambda k_\xi^\alpha n_{\sigma\tau}^{\xi\beta} \\ & + \mu l_\eta^\beta n_{\sigma\tau}^{\alpha\eta} + \lambda\mu k_\xi^\alpha l_\eta^\beta n_{\sigma\tau}^{\eta\xi} + \lambda\mu k_\sigma^\alpha l_\tau^\beta \} \bar{y}^{\sigma\tau}. \end{aligned} \tag{7}$$

A necessary and sufficient condition that the equation (4) be representable in the form (6) is that we may solve the integral equation

$$\nu M_{\sigma\tau}^{\alpha\beta} = N_{\sigma\tau}^{\alpha\beta} + \lambda N_{\xi\tau}^{\alpha\beta} K_\sigma^\xi + \mu N_{\sigma\tau}^{\alpha\beta} L_\tau^\eta + \lambda\mu N_{\xi\eta}^{\alpha\beta} K_\sigma^\xi L_\tau^\eta + \lambda\mu K_\sigma^\alpha L_\tau^\beta \tag{8}$$

for $N_{\sigma\tau}^{\alpha\beta}$. For λ and μ non-characteristic values this equation may be readily solved as has been noted in the first part of the paper. Doing this, we have

$$(9)\quad N_{\alpha\beta}^{\alpha\beta} = \nu M_{\sigma\tau}^{\alpha\beta} + \lambda\nu M_{\xi\tau}^{\alpha\beta} k_{\sigma}^{\xi} + \mu\nu M_{\sigma\eta}^{\alpha\beta} l_{\tau}^{\eta} + \lambda\mu\nu M_{\xi\eta}^{\alpha\beta} k_{\sigma}^{\xi} l_{\tau}^{\eta} - \lambda\mu k_{\sigma}^{\alpha} l_{\tau}^{\beta}.$$

THEOREM 1. *The parameters λ and μ being non-characteristic values of the kernels K_{σ}^{α} and L_{τ}^{β}, respectively, a necessary and sufficient condition that* (4) *have a unique, continuous solution is that the parameter ν have a value which does not make null the Fredholm determinant of the kernel $N_{\sigma\tau}^{\alpha\beta}$ as given in* (9). *The solution is then given by* (7).

It is to be noted that by taking the kernels of the transformations (5) in different order, that is for example

$$z^{\alpha\beta} = y^{\alpha\beta} + N_{\sigma\tau}^{\alpha\beta} y^{\sigma\tau}, \quad \bar{z}^{\alpha\beta} = z^{\alpha\beta} + \lambda K_{\sigma}^{\alpha} z^{\sigma\beta}, \quad \bar{y}^{\alpha\beta} = \bar{z}^{\alpha\beta} + \mu L_{\tau}^{\beta} \bar{z}^{\alpha\tau},$$

we shall obtain the above theorem with a different permutation of functions, but in all instances the Fredholm determinant of the kernel $N_{\sigma\tau}^{\alpha\beta}$ so obtained reduces to exactly the same quantity.

COROLLARY 1. *The parameters λ and μ being non-characteristic values of the kernels K_{σ}^{α} and L_{τ}^{β}, respectively, a necessary and sufficient condition that the equation*

$$(10)\qquad \bar{y}^{\alpha\beta} = y^{\alpha\beta} + \lambda K_{\sigma}^{\alpha} y^{\sigma\beta} + \mu L_{\tau}^{\beta} y^{\alpha\tau}$$

have a unique, continuous inverse is that the Fredholm determinant of the kernel $-\lambda\mu k_{\sigma}^{\alpha} l_{\tau}^{\beta}$ be different from zero. In addition, for λ and μ sufficiently small the solution will be given by

$$y^{\alpha\beta} = \bar{y}^{\alpha\beta} + \lambda k_{\sigma}^{\alpha} \bar{y}^{\sigma\beta} + \mu l_{\tau}^{\beta} \bar{y}^{\alpha\tau} + \{2R_{\sigma\tau}^{\alpha\beta}[-\lambda\mu k_{\rho}^{\xi} l_{\pi}^{\eta}] + \lambda k_{\xi}^{\alpha} R_{\sigma\tau}^{\xi\beta} + \mu l_{\eta}^{\beta} R_{\sigma\tau}^{\alpha\eta}\} \bar{y}^{\sigma\tau},$$

where $R_{\sigma\tau}^{\alpha\beta}[-\lambda\mu k_{\rho}^{\xi} l_{\pi}^{\eta}]$ is the resolvent kernel of $-\lambda\mu k_{\sigma}^{\alpha} l_{\tau}^{\beta}$.

To prove the last part of the corollary, we note by hypothesis that*

$$(11)\quad R_{\sigma\tau}^{\alpha\beta}[-\lambda\mu k_{\rho}^{\xi} l_{\pi}^{\eta}] = \lambda\mu k_{\sigma}^{\alpha} l_{\tau}^{\beta} + \lambda^2\mu^2 k_{\xi}^{\alpha} k_{\sigma}^{\xi} l_{\eta}^{\beta} l_{\tau}^{\eta} + \cdots,$$

and so

$$R_{\sigma\tau}^{\alpha\beta} = \lambda\mu k_{\xi}^{\alpha} l_{\eta}^{\beta} R_{\sigma\tau}^{\xi\eta} + \lambda\mu k_{\sigma}^{\alpha} l_{\tau}^{\beta}.$$

* This is the usual series development of an inverse kernel.

If the Fredholm determinant of the kernel $-\lambda\mu k_\sigma^\alpha l_\tau^\beta$ is zero, we see from (5) that the homogeneous equation (10), that is, $\bar{y}^{\alpha\beta} \equiv 0$, has a solution. Let us seek now to determine the nature of λ and μ when this condition is fulfilled. For λ and μ sufficiently small, the resolvent kernel of $-\lambda\mu k_\sigma^\alpha l_\tau^\beta$ is given by (11). To say that $D[-\lambda\mu k_\sigma^\alpha l_\tau^\beta] = 0$ is to say that (11) or

$$\overline{R}_{\sigma\tau}^{\alpha\beta} \equiv \frac{R_{\sigma\tau}^{\alpha\beta}}{\lambda\mu}$$

has a singularity for the particular values of λ and μ taken.

By the Hadamard multiplication theorem, we know that the singular values of the series

$$(12)\quad \overline{R}_{\sigma\tau}^{\alpha\beta}(z) = k_\sigma^\alpha l_\tau^\beta + \lambda\mu k_{\xi_1}^\alpha k_\sigma^{\xi_1} l_{\eta_1}^\beta l_\tau^{\eta_1} z + \cdots + \lambda^n \mu^n k_{\xi_1}^\alpha k_{\xi_2}^{\xi_1} \cdots k_\sigma^{\xi_n} l_{\eta_1}^\beta l_{\eta_2}^{\eta_1} \cdots l_\tau^{\eta_n} z^n + \cdots$$

will be given by the product of the singularities of the two series

$$(13)\quad k_\sigma^\alpha(z) = \sum_{i=0}^{\infty} \lambda^i k_{\xi_1}^\alpha k_{\xi_2}^{\xi_1} \cdots k_\sigma^{\xi_i} z^i, \quad l_\tau^\beta(z) = \sum_{i=0}^{\infty} \mu^i l_{\eta_1}^\beta l_{\eta_2}^{\eta_1} \cdots l_\tau^{\eta_i} z^i.$$

By means of a well known relation satisfied by a resolvent kernel, namely

$$\frac{d^n}{d\lambda^n}\{k_\sigma^\alpha\} = n! k_{\xi_1}^\alpha k_{\xi_1}^{\xi_1} \cdots k^{\xi_n},$$

we see that the equations (13) reduce to

$$(14)\quad k_\sigma^\alpha(z) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \frac{d^i}{d\lambda^i}\{k_\sigma^\alpha\} z^i, \; l_\tau^\beta(z) = \sum_{i=0}^{\infty} \frac{\mu^i}{i!} \frac{d^i}{d\mu^i}\{l_\tau^\beta\} z^i.$$

Since the equations (14) are of the form of a MacLaurin's expansion, it is easily seen that the singular ensembles of $k_\sigma^\alpha(z)$ and $l_\tau^\beta(z)$ are given respectively by $z = \bar{\lambda}/\lambda - 1$ and $z = \bar{\mu}/\mu - 1$, where $\bar{\lambda}$ and $\bar{\mu}$ are characteristic values of the kernels K_σ^α and L_τ^β, respectively. Thus by Hadamard's theorem the singular ensemble of (12) is given by $z = (\bar{\lambda}/\lambda - 1)(\bar{\mu}/\mu - 1)$. If we define $z = 1$ to belong to this ensemble, then we see that λ and μ must satisfy the relationship $\lambda/\bar{\lambda} + \mu/\bar{\mu} = 1$.

THEOREM 2. *For λ and μ sufficiently small and non-characteristic values of their respective kernels, a necessary and sufficient condition that the equation*

$$y^{\alpha\beta} + \lambda K_\sigma^\alpha y^{\sigma\beta} + \mu L_\tau^\beta y^{\alpha\tau} = 0 \tag{15}$$

have a homogeneous solution is that the parameters satisfy a relation

$$\frac{\lambda}{\bar{\lambda}} + \frac{\mu}{\bar{\mu}} = 1, \tag{16}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are characteristic values of the kernels K_σ^α and L_τ^β, respectively.

The sufficiency of the theorem is evident since (16) implies $\lambda\mu = (\bar{\lambda}-\lambda)(\bar{\mu}-\mu)$ and this in (12) is singular because of (14).

Let us now consider the ordinary associate equations with the kernels K_σ^α and L_τ^β respectively, that is,

$$\begin{cases} \Psi_\alpha^i + \bar{\lambda} K_\alpha^\sigma \Psi_\sigma^i = 0, & (i = 1, 2, \cdots, n), \\ {}'\Psi_\beta^j + \bar{\mu} L_\beta^{\tau\prime} \Psi_\tau^j = 0, & (j = 1, 2, \cdots, m). \end{cases} \tag{17}$$

These two equations have the solutions as indicated. In multiplying (15) separately by $\bar{\lambda}\Psi_\alpha^i$ and $\bar{\mu}'\Psi_\beta^j$ and integrating, we obtain the equations

$$\bar{\lambda}\Psi_\alpha^i y^{\alpha\beta} + \lambda\bar{\lambda}\Psi_\alpha^i K_\sigma^\alpha y^{\sigma\beta} + \bar{\lambda}\mu\Psi_\alpha^i L_\tau^\beta y^{\alpha\tau} = 0,$$
$$\bar{\mu}'\Psi_\beta^j y^{\alpha\beta} + \lambda\bar{\mu}'\Psi_\beta^j K_\sigma^\alpha y^{\sigma\beta} + \mu\bar{\mu}'\Psi_\beta^j L_\tau^\beta y^{\alpha\tau} = 0.$$

These equations in virtue of (17) and (16) reduce to

$$\Psi_\alpha^i \left\{ y^{\alpha\beta} + \bar{\mu} L_\tau^\beta y^{\alpha\tau} \right\} = 0, \quad {}'\Psi_\beta^j \left\{ y^{\alpha\beta} + \bar{\lambda} K_\sigma^\alpha y^{\sigma\beta} \right\} = 0. \tag{18}$$

It is evident that all solutions of (15) must satisfy simultaneously the equations (18); hence we have the following theorem.

THEOREM 3. *All solutions of* (15), *for a particular set of characteristic parameters $\bar{\lambda}$ and $\bar{\mu}$ satisfying* (16), *are of the form*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \Phi_i^\alpha {}'\Phi_j^\beta,$$

where the c_{ij} are constants, and Φ_i^α and ${}'\Phi_j^\beta$ are the respective fundamental solutions of the homogeneous equations

$$\phi^\alpha + \bar{\lambda} K_\sigma^\alpha \phi^\sigma = 0, \quad {}'\phi^\beta + \bar{\mu} L_\tau^\beta {}'\phi^\tau = 0.$$

With respect to the above theorem we must bear in mind that for an equation (15) with given parameter values λ and μ, there may exist more than one pair of characteristic parameters satisfying (16) and accordingly the totality of homogeneous solutions of (15) is enlarged.

It follows directly from the above theorem that the totality of solutions of the *associate* equation to (15), that is,

$$y_{\alpha\beta} + \lambda K_\alpha^{\ \sigma} y_{\sigma\beta} + \mu L_\beta^{\ \tau} y_{\alpha\tau} = 0, \tag{19}$$

have the form

$$\sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} \Psi_\alpha^i {}' \Psi_\beta^j .$$

We have already noted the equivalence between (15) and the equation

$$y^{\alpha\beta} - \lambda\mu k_\sigma^\alpha l_\tau^\beta y^{\sigma\tau} = 0. \tag{20}$$

In like manner, we may show the equivalence between (19) and

$$y_{\alpha\beta} - \lambda\mu k_\alpha^\sigma l_\beta^\tau y_{\sigma\tau} = 0. \tag{21}$$

Since λ and μ are not characteristic values of their respective kernels, by adding $\lambda\mu K_\sigma^\alpha L_\tau^\beta y^{\sigma\tau}$ to (10) and applying (3), we may write equation (10) in the form

$$y^{\alpha\beta} - \lambda\mu k_\sigma^\alpha l_\tau^\beta y^{\sigma\tau} = \bar{y}_\sigma^{\alpha\beta} + \lambda k_\sigma^\alpha \bar{y}^{\sigma\beta} + \mu l_\tau^\beta \bar{y}^{\alpha\tau} + \lambda\mu k_\sigma^\alpha l_\tau^\beta \bar{y}^{\sigma\tau}. \tag{22}$$

A necessary and sufficient condition that (22) have a solution when λ and μ satisfy (16) is that all solutions of (21) when composed with the right hand side of (22) yield zero. Furthermore, from the identities

$$\Psi_\sigma^i k_\alpha^\sigma = \frac{1}{\bar{\lambda} - \lambda} \Psi_\alpha^i, \quad {}' \Psi_\sigma^j l_\alpha^\sigma = \frac{1}{\bar{\mu} - \mu} {}' \Psi_\alpha^j ,$$

we see that *a necessary and sufficient condition that the equation* (10) *have a solution when the parameters satisfy a relationship* (16) *is that the function* $\bar{y}^{\alpha\beta}$ *satisfy the* $n \cdot m$ *equations*

$$\Psi_\alpha^i {}' \Psi_\beta^j \bar{y}^{\alpha\beta} = 0, \qquad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, m).$$

The actual solutions may be computed from the equation (22) in the usual fashion.

LEMMA. *A necessary and sufficient condition that*

$$A^{\alpha\beta}y^{\alpha\beta} + B^{\alpha\beta}_{\sigma}y^{\sigma\beta} + C^{\alpha\beta}_{\tau}y^{\sigma\tau} + D^{\alpha\beta}_{\sigma\tau}y^{\sigma\tau} = 0$$

be true for all continuous functions $y^{\alpha\beta}$ *is that*

$$A^{\alpha\beta} \equiv 0,\ B^{\alpha\beta}_{\sigma} \equiv 0,\ C^{\alpha\beta}_{\tau} \equiv 0,\ D^{\alpha\beta}_{\tau\sigma} \equiv 0$$

in the field of continuous functions.

This lemma may be easily verified by taking $y^{\alpha\beta}=\phi^{\alpha}\psi^{\beta}$ and applying repeatedly a known lemma* that if

$$P^{\alpha}f^{\alpha} + Q^{\beta}_{\sigma}f^{\sigma} = 0$$

be true for all continuous functions f^{α}, then $P^{\alpha}\equiv 0$, $Q^{\alpha}_{\sigma}\equiv 0$.

In virtue of the transitivity of equations of the form (4) and the preceding lemma, we see that a necessary and sufficient condition that

$$y^{\alpha\beta} = \bar{y}^{\alpha\beta} + \lambda k^{\alpha}_{\sigma}\bar{y}^{\sigma\beta} + \mu l^{\beta}_{\tau}\bar{y}^{\alpha\tau} + m^{\alpha\beta}_{\tau\sigma}\bar{y}^{\sigma\tau},$$

where we have assumed λ and μ to have non-characteristic values, be an inverse to (4) is that the kernel $M^{\alpha\beta}_{\sigma\tau}$ satisfy the relations

$$(23)\quad \begin{cases} m^{\alpha\beta}_{\sigma\tau} + \nu m^{\alpha\beta}_{\sigma\tau} + \nu m^{\alpha\beta}_{\xi\eta}M^{\xi\eta}_{\sigma\tau} + \lambda\nu k^{\alpha}_{\xi}M^{\xi\beta}_{\sigma\tau} + \mu\nu l^{\beta}_{\eta}M^{\alpha\eta}_{\sigma\tau} + \lambda m^{\alpha\beta}_{\xi\tau}K^{\xi}_{\sigma} \\ \qquad + \mu m^{\alpha\beta}_{\sigma\eta}L^{\eta}_{\tau} + \lambda\mu k^{\alpha}_{\sigma}L^{\beta}_{\tau} + \lambda\mu K^{\alpha}_{\sigma}l^{\beta}_{\tau} = 0, \\ m^{\alpha\beta}_{\sigma\tau} + \nu m^{\alpha\beta}_{\sigma\tau} + \nu m^{\alpha\beta}_{\xi\eta}M^{\xi\eta}_{\sigma\tau} + \lambda\nu M^{\alpha\beta}_{\xi\tau}k^{\xi}_{\sigma} + \mu\nu M^{\alpha\beta}_{\sigma\eta}l^{\eta}_{\tau} + \lambda K^{\alpha}_{\xi}m^{\xi\beta}_{\sigma\tau} \\ \qquad + \mu L^{\beta}_{\eta}m^{\alpha\eta}_{\sigma\tau} + \lambda\mu k^{\alpha}_{\sigma}L^{\beta}_{\tau} + \lambda\mu K^{\alpha}_{\sigma}l^{\beta}_{\tau} = 0. \end{cases}$$

THEOREM 4. *The parameters* λ *and* μ *being non-characteristic and satisfying no relationship* (16), *a necessary and sufficient condition that a linear integral equation of the form*

$$\bar{y}^{\alpha\beta} = y^{\alpha\beta} + \lambda K^{\alpha}_{\sigma}y^{\sigma\beta} + \mu L^{\beta}_{\tau}y^{\alpha\tau}$$

have a solution in the same form is that the kernels K^{α}_{σ} *and* L^{β}_{τ} *satisfy conditions*

$$(24)\quad cK^{\alpha}_{\sigma} + K^{\alpha}_{\sigma} + c\lambda K^{\alpha}_{\xi}K^{\xi}_{\sigma} = 0, \qquad cL^{\beta}_{\tau} - L^{\beta}_{\tau} + c\mu L^{\beta}_{\eta}L^{\eta}_{\tau} = 0,$$

* See Michal and Peterson, *The invariantive theory of functional forms under the group of linear functional transformations of the third kind*, Annals of Mathematics, vol. 32 (1932), p. 432.

simultaneously, where c is a constant.

From (23) we see by hypothesis that we must have

$$k_\sigma^\alpha L_\tau^\beta + K_\sigma^\alpha l_\tau^\beta = 0.$$

Dividing this equation by $K_\sigma^\alpha L_\tau^\beta$, we see that it is necessary that

$$k_\sigma^\alpha = cK_\sigma^\alpha, \quad l_\tau^\beta = -cL_\tau^\beta,$$

where c is a constant. Substituting these values of the resolvent kernels in the well known relation satisfied by a kernel and its resolvent, we obtain (24). The sufficiency of the theorem is easily shown by noting that the hypotheses of the theorem imply from (23) that

$$m_{\sigma\tau}^{\alpha\beta} + \lambda K_\xi^\alpha m_{\sigma\tau}^{\xi\beta} + \mu L_\eta^\beta m_{\sigma\tau}^{\alpha\eta} = 0.$$

Since λ and μ satisfy no relation of the form (16), by Corollary 1, it follows that $m_{\sigma\tau}^{\alpha\beta} \equiv 0$.

As an example of kernels whose resolvent kernel is just a constant times the kernel itself, consider the kernel $K_\sigma^\alpha = A^\alpha B_\sigma$; then $k_\sigma^\alpha = -K_\sigma^\alpha/(1+K_\xi^\xi)$. If the kernels K_σ^α and L_τ^β of the above theorem are of this type, then it is necessary that

$$K_\sigma^\sigma + L_\sigma^\sigma = -2,$$

in order that the theorem apply.

In concluding this paper, it is of interest to note that the linear integral equation

$$\bar{y}^{\alpha\beta} = A^{\alpha\beta} y^{\alpha\beta} + \lambda B_\sigma^{\alpha\beta} y^{\sigma\beta} + \mu C^{\alpha\beta} y^{\alpha\tau} + \nu D_{\tau\sigma}^{\alpha\beta} y^{\sigma\tau} \tag{25}$$

may be reduced to the form (4) provided the kernels of (25) are of such a nature that $B_\sigma^{\alpha\beta}/A^{\alpha\beta}$ is free of β, $C_\tau^{\alpha\beta}/A^{\alpha\beta}$ is free of α, and $A^{\alpha\beta} \neq 0$, $(a \leqq \alpha, \beta \leqq b)$.

LOS ANGELES, CALIFORNIA

SOLUTION OF HUNTINGTON'S "UNSOLVED PROBLEM IN BOOLEAN ALGEBRA"

BY A. A. BENNETT

The sixth set of postulates for Boolean algebra recently proposed by E. V. Huntington* may, as he suggests, be modified so as to read as follows. Let

K = an undefined class containing at least two elements, $a, b, c, \cdots$;

T = an undefined subclass in K (so that if a is in T, then a is in K);

$(a+b)$ = the result of an undefined binary operation; and

a' = the result of an undefined unary operation.

POSTULATE 1.71. If a and b are in K, then $(a+b)$ is in K.

POSTULATE 1.7. If a is in K, then a' is in K.

POSTULATE 1.1. If a is in T and $(a'+b)$ is in T, then b is in T.

POSTULATE 1.2. If a is in K, then $[(a+a)'+a]$ is in T.

POSTULATE 1.3. If a, b, etc. are in K, then $[b'+(a+b)]$ is in T.

POSTULATE 1.4. If a, b, etc. are in K, then $[(a+b)'+(b+a)]$ is in T.

POSTULATE 1.6. If a, b, c, etc. are in K, then $\{(b'+c)'+[(a+b)'+(a+c)]\}$ is in T.

POSTULATE 1.8. If $(a'+b)$ is in T and $(b'+a)$ is in T, then $a=b$.

The "unsolved problem" he proposes is the question whether or not Postulate 1.1 is independent of the other postulates in this list. The purpose of the present paper is to answer this question

* E. V. Huntington, *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Transactions of this Society, vol. 35 (1933), pp. 274–304, especially p. 298. Huntington's sixth set, while inferior to his fourth set when regarded merely as a set of postulates for Boolean algebra, is of interest in connection with B. A. Bernstein's version of the primitive propositions of the *Principia* (see the bibliography in the paper cited). In connection with Huntington's fourth set, it should be noted that Postulate 4.5 is not independent; see the forthcoming number of the Transactions of this Society.

in the affirmative by constructing an actual example of a system $(K, T, +, ')$ which satisfies all the other postulates of the list, but not Postulate 1.1.

For this example, $+$ is nonassociative but is such that I. *If a is in K, then* $(a+a)=a$, and II. *If a and b are in K, then* $(a+b)=(b+a)$. The distinct elements of K are here countably infinite. Each has a unique *rank* equal to the sum of the minimum number of signs, $+$, and $'$, required to represent the element. In particular there is a unique undefined element, e, of rank zero. To obtain all the distinct elements of rank, n, >0, one proceeds as follows by recursion. First, for each previously recorded element a of rank $n-1$, write a'. Secondly, for each element a of rank r, where $(n/2) \leqq r < n$, and for each element, b, of rank $n-r-1$, write $(a+b)$. Thirdly, if n is odd (say $n=2m+1$), for each element a of rank m (other than the last recorded element of rank m) and for each element b of rank m occurring subsequent to a in the recorded list, write $(a+b)$. Thus e' is the only element of rank 1. The distinct elements of rank 2 are e'', $(e'+e)$. Those of rank 3 are e''', $(e'+e)'$, $(e''+e)$, $[(e'+e)+e]$. Similarly for elements of higher rank.

From the method of construction, it is evident that Postulate 1.7 is satisfied, and that for $a \neq b$, Postulate 1.71 is satisfied. By virtue of I, II, this latter postulate holds also for $a=b$.

The subclass T will be defined for this example as consisting exclusively of all elements of K each of which is of one of the four following types, A-D: A, the unique element e; B, $(a'+a)$; C, $[b'+(a+b)]$, for $b \neq a$; D, $\{(b'+c)'+[(a+b)'+(a+c)]\}$, for a, b, c, not all equal. It follows that Postulates 1.2, 1.3, 1.4, 1.6, are satisfied. Indeed Postulate 1.2 reduces by I to B. Postulate 1.3 coincides with C if $b \neq a$, but for $b=a$, reduces by I to B. Postulate 1.4 reduces by II to a special case of B. Postulate 1.6 coincides with D save for $a=b=c$, in which case by use of I it reduces to a special case of B.

From the method of construction of the set of elements of the system one has the following theorem.

III. *If a and b are in K, then $a=b$ if and only if a and b are reducible to identical form in $(e, +, ')$ by at most repeated use of* I *and* II *alone.*

It remains only to show that in this example Postulate 1.8 is satisfied and Postulate 1.1 is violated.

The explicit details required for rigorous proof involve much repetition unless one makes use of lemmas (here denoted by Arabic numerals). While III is frequently invoked, it is needed formally only in certain more specialized forms here listed as Lemmas 1–8, together sufficient to replace III in this discussion. Each of these follows from III by inspection, by the simple expedient of comparing the minimum number of times a sign + or ′ appears in formal expressions for elements being compared.

1. (i) If $(a+b)=b$, then $a=b$. (ii) If $[a+(b+c)]=c$, then $a=b=c$.

2. If $a\neq b$, and if $c\neq d$, and if $(a+b)=(c+d)$, then either (i) $a=c$ and $b=d$, or else (ii) $a=d$ and $b=c$.

3. a, a', a'', a''', etc., are distinct.

4. If $a'=b'$, then $a=b$.

5. (i) $(a+b')\neq b$, (ii) $(a+b)'\neq b$, (iii) $[(a+b)'+c]\neq b$, (iv) $(a+b')'\neq b$, (v) $[(a+b)'+c]'\neq b$, (vi) $(a+b'')\neq b$.

6. (i) $(u''+u)\neq(a'+a)$, (ii) $(u'''+u)\neq(a'+a)$.

7. If $(a+b)=c'$, then $a=b=c'$.

8. (i) $a'\neq e$, (ii) $(a'+b)\neq e$.

We now prove Lemmas 9–19, using I and II and Lemmas 1–8.

9. (i) $(a+b)\neq b'$, since otherwise by 7, $a=b=b'$, violating 3. (ii) $(a+b)'\neq(a+c)$, since otherwise by 7, $a=(a+b)'$ violating 5(ii), using II. (iii) $(a+b)\neq b'''$, since otherwise by 7, $a=b=b'''$, violating 3.

10. If $c=[(a+b)'+(a+b')]$, then (i) $c\neq a$, (ii) $c\neq b$, (iii) $c'\neq b$, (iv) $c\neq b'$.

Proof. (i) $[(a+b)'+(a+b')]\neq a$ by 5 (iii). (ii) $[(a+b)'+(a+b')]\neq b$, by 5 (iii). (iii) $[(a+b)'+(a+b')]'\neq b$, by 5 (v). (iv) $[(a+b)'+(a+b')]\neq b'$, since otherwise by 7, $(a+b)'=(a+b')$, which violates 9 (ii).

11. $[(a+b)'+(a+c)]\neq d'$, since otherwise by 7, $(a+b)'=(a+c)$, violating 9 (ii).

12. $[(a+b)'+(a+c)]\neq c$, since otherwise by 1 (ii), $(a+b)'=a$, violating 5 (ii).

13. If $\{[(a+b)'+(a+c)]+d\}=(b'+c)$, then $a=b=c$, and hence $d=(a'+a)$.

Proof. A. Let $c=b'$. Then $\{[(a+b)'+(a+b')]+d\}=b'$, and by 7, $[(a+b)'+(a+b')]=b'$, and again by 7, $(a+b)'=(a+b')=b'$. From $(a+b)'=b'$, by 4 and 1 (i) follows $a=b$.

From $(a+b')=b'$, by 1 (i) or 7 follows $a=b'$. Then $b=b'$ contrary to 3.

B. Let $c\neq b'$, and $d\neq[(a+b)'+(a+c)]$. Then by 2, either (i) $b'=[(a+b)'+(a+c)]$ and $d=c$, or else (ii) $b'=d$, and $c=[(a+b)'+(a+c)]$. But (i) violates 11, and (ii) violates 12.

C. Let $c\neq b'$, but $d=[(a+b)'+(a+c)]$. Then by I, 2 and 4 either (i) $b=(a+b)$ and $c=(a+c)$ or else (ii) $b'=(a+c)$ and $c=(a+b)'$. In (i) by 1(i), $a=b$ and $a=c$. Hence $a=b=c$, as allowed for in the hypothesis of the theorem. In (ii) by 7, $a=c=b'$, and on substituting in $c=(a+b)'$ one has $b'=(b'+b)'$, whence by 4, $b=(b'+b)$, which by 1(i) violates 3.

14. $[(a+b)'+(a+c)]'\neq(b'+c)$, since otherwise by 7, $b'=c=[(a+b)'+(a+c)]'$, and by 4, $b=[(a+b)'+(a+c)]$, violating 5 (iii).

15. (i) $(u'''+u)\neq[b'+(a+b)]$, (ii) $(u'''+u)\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$.

PROOF. (i) By 3 and 9 (i) one may apply 2. Under one alternative, $u'''=b'$ and $u=(a+b)$. Hence by 4, one has $u=(a+u'')$ contradicting 5(vi). The other alternative yields $u'''=(a+b)$, $u=b'$. Hence, by 7, $a=b=u'''$, or $u=u^{iv}$, contradicting 3. (ii) By 3, 11, 2 and 4, either $u''=(b'+c)$ and $u=[(a+b)'+(a+c)]$ or else $u'''=[(a+b)'+(a+c)]$ and $u=(b'+c)'$. In the former case by 7, $b'=c=u''$, and by 4, $b=u'$, whence $[(a+b)'+(a+c)]'=b$ contrary to 5(v). In the latter case by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

16. If $[(x+y)'+(x+z)]=(b'+c)$ and also $[(a+b)'+(a+c)]=(y'+z)$, then $a=b=c=x=y=z$.

PROOF. By 11, $c\neq b'$ and $z\neq y'$. By 9(ii), $(x+y)'\neq(x+z)$, and $(a+b)'\neq(a+c)$. Hence, by 2 and 4, one of the four following conditions holds. (i) $(x+y)=b$, $(x+z)=c$, $(a+b)=y$, $(a+c)=z$, or (ii) $(x+y)=b$, $(x+z)=c$, $(a+b)'=z$, $(a+c)=y'$, or (iii) $(x+y)'=c$, $(x+z)=b'$, $(a+b)=y$, $(a+c)=z$, or (iv) $(x+y)'=c$, $(x+z)=b'$, $(a+b)'=z$, $(a+c)=y'$. In (i) we have $[x+(a+b)]=b$, $[x+(a+c)]=c$. Hence by 1(ii), $x=a=b=c$, and upon substituting, $a=b=c=x=y=z$. In (ii), upon substituting one has $[a+(x+y)]'=z$, and $[a+(x+z)]=y'$. Hence by 7, $a=(x+z)=y'$ and hence by 7 again, $a=x=z=y'$. Hence $[y'+(y'+y)]'=y'$, or by 4, $[y'+(y'+y)]=y$ so that by 1(ii), $y=y'$ contradicting 3. Case (iii) differs from case (ii) only by a change of letters throughout. In case (iv) by 7, one has $x=z=b'$, $a=c=y'$.

Hence upon substituting, $(x+y)'=y'$, $(a+b)'=b'$. Hence by 4 and 1(i), $x=y$, $a=b$. But then $b=y'$ and $y=b'$, or $b=b''$ contrary to 3.

17. d' is not in T.

PROOF. A. $d'\neq e$ (by 8(i)); B. $d'\neq(a'+a)$, (by 7 and 3); C. $d'\neq[b'+(a+b)]$ for $a\neq b$, (by 7 and 9(i)); D. $d'\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$, where a, b, c are not all equal, for otherwise by 7, $(b'+c)'=[(a+b)'+(a+c)]$, and by 7 again, $(a+b)'=(a+c)$ contrary to 9(ii).

18. If x and y are distinct elements in K, then $[(x+y)'+y]$ is not in T.

PROOF. A. $[(x+y)'+y]\neq e$, by 8(ii).

B. $[(x+y)'+y]\neq(a'+a)$ for $x\neq y$, since otherwise by 3, 5(ii), 2, and 4, either (i) $(x+y)=a$ and $y=a$, or else (ii) $(x+y)'=a$, and $y=a'$. But in (i) $(x+y)=y$, and hence by 1(i), $x=y$, contrary to hypothesis. In (ii) $(x+a')'=a$, contrary to 5(iv).

C. For $x\neq y$, $[(x+y)'+y]\neq[b'+(a+b)]$, for $a\neq b$. Otherwise by 2 and 4, either (i) $(x+y)=b$ and $y=(a+b)$, or (ii) $(x+y)'=(a+b)$ and $y=b'$. In (i) $[x+(a+b)]=b$, and hence by 1(ii) $x=a=b$, and hence also $x=y$, contrary to hypothesis. In (ii) by 7, $a=b=(x+y)'$, and hence $b=(x+b')'$, contrary to 5(iv).

D. For $x\neq y$, $[(x+y)'+y]\neq\{(b'+c)'+[(a+b)'+(a+c)]\}$, for a, b, c not all equal.

PROOF. By 11, $[(a+b)'+(a+c)]\neq(b'+c)'$, and by 5(i), $y\neq(x+y)'$. Hence were the theorem false in this case, by 2 and 4, either (i) $(x+y)=(b'+c)$ and $y=[(a+b)'+(a+c)]$, or else (ii) $(x+y)'=[(a+b)'+(a+c)]$ and $y=(b'+c)'$. In (i) $\{x+[(a+b)'+(a+c)]\}=(b'+c)$. Hence by 13, $a=b=c$, and $x=(a'+a)=y$, contrary to hypothesis. In (ii) by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

19. $\{[(x+y)'+(x+z)]'+(y'+z)\}$ is not in T if x, y, z are not all equal.

PROOF. A. $\{[(x+y)'+(x+z)]'+(y'+z)\}\neq e$, by 8(ii).

B. $\{[(x+y)'+(x+z)]'+(y'+z)\}\neq(a'+a)$, unless $x=y=z$. By 14, 3, 2, and 4, either (i) $[(x+y)'+(x+z)]=a$ and $(y'+z)=a$, or (ii) $[(x+y)'+(x+z)]'=a$, and $(y'+z)=a'$. In (i), $[(x+y)'+(x+z)]=(y'+z)$ which by 13 (for $d=[(x+y)'+(x+z)]$) is only possible for $x=y=z$. In (ii), $[(x+y)'+(x+z)]''=(y'+z)$.

Hence by 7, $z=y'=[(x+y)'+(x+z)]''$ and by 4, $y=[(x+y)'+(x+z)]'$, contradicting 5(v).

C. For x, y, z not all equal, $\{[(x+y)'+(x+z)]'+(y'+z)\} \neq [b'+(a+b)]$, where $a\neq b$. By 14, 9(i), 2 and 4, either (i) $[(x+y)'+(x+z)]=b$ and $(y'+z)=(a+b)$, or else (ii) $[(x+y)'+(x+z)]'=(a+b)$ and $(y'+z)=b'$. In (i) $(y'+z)=\{a+[(x+y)'+(x+z)]\}$ which by 13 is possible only for $x=y=z$. In (ii), by 7 and 4, $a=b=[(x+y)'+(x+z)]'$, $y=b$, $z=y'$. Hence upon substituting, $[(x+y)'+(x+y')]'=y$ contradicting 10(iii).

D. For x,y,z not all equal, $\{[(x+y)'+(x+z)]'+(y'+z)\} \neq \{(b'+c)'+[(a+b)'+(a+c)]\}$, where a, b, c, are not all equal. Otherwise by 14, 11, 2 and 4, either (i) $[(x+y)'+(x+z)]=(b'+c)$ and $(y'+z)=[(a+b)'+(a+c)]$, or else (ii) $[(x+y)'+(x+z)]'=[(a+b)'+(a+c)]$ and $(y'+z)=(b'+c)'$. In case (i) by 16, $a=b=c=x=y=z$, excluded by hypothesis. In case (ii) by 7, $(a+b)'=(a+c)$ contrary to 9(ii).

THEOREM. *Postulate* 1.8 *is satisfied.*

PROOF. The postulate may be restated as follows: *If* $u\neq v$, *and if* $(u'+v)$ *is in* T, *then* $(v'+u)$ *is not in* T. To prove this one need only test the possible expressions for $(u'+v)$, with $u\neq v$, which are in T. One may note first that $v\neq u'$, since by 17, u' is not in T. Consider in turn the available alternatives A–D. A. $(u'+v)\neq e$, by 8(ii). B. Let $(u'+v)=(a'+a)$. Since $v\neq u'$, one has by 2 and 4, either $u=a=v$, (contrary to hypothesis), or else $u'=a$ and $v=a'$, so that $(v'+u)=(u'''+u)$. But the possible alternatives A–D for this subcase are to be successively rejected by use of 8(ii), 6(ii), 15(i) and (ii). C. Let $(u'+v)=[b'+(a+b)]$, for $a\neq b$. Then by 2 and 4, either (i) $u=b$, and $v=(a+b)$ or (ii) $u'=(a+b)$ and $v=b'$. In (i) $(v'+u)=[(a+b)'+b]$, with $a\neq b$. But by 18, this is not in T. In (ii), by 7, $a=b$, contrary to hypothesis. D. Let $(u'+v)=\{(b'+c)'+[(a+b)'+(a+c)]\}$, with $u\neq v$, and with a, b, c not all equal. Then in view of 17, by 2 and 4, it follows that either (i) $u=(b'+c)$ and $v=[(a+b)'+(a+c)]$ or else (ii) $u'=[(a+b)'+(a+c)]$ and $v=(b'+c)'$. In (i) $(v'+u)$ cannot be in T by 19 while (ii) contradicts 11.

THEOREM. *Postulate* 1.1 *fails to hold.*

PROOF. e is in T, (A), and hence, (C), $[e'+(e''+e)]$ is in T. But as is now to be shown $(e''+e)$ is not in T. Indeed, we have

A. $(e''+e) \neq e$, by 8(ii). B. $(e''+e) \neq (a'+a)$, by 6(i). C. $(e''+e) \neq [b'+(a+b)]$. For otherwise, by 3, 9(i), 2 and 4, either (i) $e'=b$ and $e=(a+b)$, or else (ii) $e=b'$ and $e''=(a+b)$. But (i) is impossible since $(a+b)' \neq b$ by 5(ii), and (ii) is impossible since $e \neq b'$ by 8(i). D. $(e''+e) \neq \{(b'+c)'+[(a+b)'+(a+c)]\}$. Indeed otherwise in view of 3, 11, 2 and 4, either (i) $e'=(b'+c)$ and $e=[(a+b)'+(a+c)]$ which contradicts 8(ii), or else (ii) $e''=[(a+b)'+(a+c)]$ and $e=(b'+c)'$ which contradicts 8(i) and also 11.

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CONCURRENCE AND UNCOUNTABILITY*

BY N. E. RUTT

1. *Introduction.* The point set of chief interest in this paper, a plane bounded continuum Z, is the sum of a continuum X and a class of connected sets $[X_\alpha]$, each element X_α of which has at least one limit point in X and is a closed subset of $c_u(X+X_b)$, where X_b is any element of $[X_\alpha]$ different from X_a and where $c_u(X+X_b)$ is the unbounded component of the plane complement of the set $X+X_b$. Upon a basis of separation properties, order† may be assigned to the elements of $[X_\alpha]$ agreeing in its details with that of some subset of a simple closed curve. We shall use some definite element X_r of $[X_\alpha]$ as reference element, selecting as X_r one of $[X_\alpha]$ containing a point arcwise accessible from $c_u(Z)$. A countable subcollection $[X_i^k]$ of $[X_\alpha]$ excluding X_r is called a *series* if for each j, $(j=2, 3, 4, \cdots)$, the elements X_j and X_r separate X_{j-1} and X_{j+1}. Two different series $[X_i^k]$ and $[X_i^k]$ are said to be *opposite in sense* if there exist different subscripts m and n such that X_m^k and X_m^k separate both X_n^k and X_n^k from X_r; otherwise they are said to have the *same sense*. They are said to be *concurrent* if they have the same sense and if there exists no element of $[X_\alpha]$ which together

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† R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202; J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562; N. E. Rutt, *On certain types of plane continua*, Transactions of this Society, vol. 33, No. 3, pp. 806–816.

with X_r separates infinitely many of the one from infinitely many of the other. It is easily seen that two series with infinitely many elements in common are concurrent, that two non-concurrent series may exist such that no element of $[X_\alpha]$ separates infinitely many of one from infinitely many of the other with respect to X_r, and that when two series having the same sense are not concurrent, then one of the two contains an element which together with X_r separates all the elements of one series from all but a finite number of the elements of the other. This paper deals mainly with collections $[[X_i]_\alpha]$ of such series as $[X_i^k]$. Sets whose elements are series of this sort have some properties which are close analogs of properties of the collection $[X_\alpha]$. For instance, when no two of four given elements of $[[X_i]_\alpha]$ are concurrent, then some pair of the four will separate the other pair in a sense easily distinguished.

THEOREM 1. *If* $[[X_i]_\alpha]$ *is a collection of series of* $[X_\alpha]$, *no two of which are opposite in sense and no two of which are concurrent, then* $[[X_i]_\alpha]$ *is not both well-ordered and uncountable.*

If we suppose otherwise, we arrive at a contradiction. Consider the element $[X_i]_\lambda$ of the well-ordered collection $[[X_i]_\alpha]$. If λ is any transfinite ordinal of the type $\mu+n$, where n is a positive integer and μ is a transfinite ordinal not of this type, then $[X_i]_\lambda$ contains an element X_{i_λ} which, together with X_r, separates all the elements of every series $[X_i]_\rho$ of $[[X_i]_\alpha]$ having ρ preceding λ from every element X_j of $[X_i]_\lambda$ having $j>i_\lambda$. Let $[Y_i]_\lambda$ be that series of $[X_\alpha]$ whose elements are the elements of $[X_i]_\lambda$ that have subscripts greater than i_λ. Let $[[Y_i]_\beta]$ be the collection of all series $[Y_i]_\beta$ which may be obtained from such elements of $[[X_i]_\alpha]$ as $[X_i]_\lambda$ with λ as specified above. The collection $[[Y_i]_\beta]$ is well-ordered, and consists of elements no two of which are concurrent or opposite in sense. That it is also uncountable may be seen directly from the fact that following immediately every element in $[[X_i]_\alpha]$ which does not contribute an element to $[[Y_i]_\beta]$ is one which does. Accordingly $[X_\beta]$, that maximal subcollection of $[X_\alpha]$ each element of which is included in an element of $[[X_i]_\beta]$, is likewise well-ordered and uncountable. Now $[X_\beta]$ includes an uncountable subcollection $[Y_\beta]$, likewise well-ordered, each element of which contains a point at a distance from X greater than some

definite positive quantity ϵ. Let C be a simple closed curve enclosing X whose interior contains no point at a distance greater than $\epsilon/2$ from some point of X. Each element of $[Y_\beta]$ has points within and without C. For each element Y_b of $[Y_\beta]$ let y_b be a point of C which is limit of that component of the subset of $X+Y_b$ interior to C which contains X. The collection $[y_b]$ is both well-ordered* and uncountable, which constitutes a contradiction.†

2. THEOREM 2. *If $[[X_i]_a]$ is a collection of series, no two of which are concurrent or opposite in sense, and for every element $[X_i]_a$ of $[[X_i]_a]$ there is a point a of the element X_a of $[X_a]$ different from X_r within every neighborhood of which there is a point of some element of $[X_i]_a$, then $[[X_i]_a]$ is well-ordered.*

It will be shown that, given any element $[X_i]_a$ of $[[X_i]_a]$, there is a first element of $[[X_i]_a]$, infinitely many of whose elements separate all of $[X_i]_a$ from X_a with respect to X_r. If we suppose that there is none, a contradiction will be obtained. This supposition implies that $[[X_i]_a]$ includes a countable sequence $[[X_i]_j]$ such that, owing to the fact that in every one of $[[X_i]_a]$, when $m>n$, then the mth element separates the nth from X_a with respect to X_r‡, for $j=1, 2, 3, \cdots$, all but at most a finite number of $[X_i]_{j+1}$ separate all of $[X_i]_a$ from X_a and from all but a finite number of $[X_i]_j$. It would thus be allowable to suppose that all of $[X_i]_{j+1}$ separate all of $[X_i]_a$ from all of $[X_i]_j$; and, for simplification, this will be assumed. Consider the prime ends,§ a simple closed curve C of $c_u(X_r+X+X_a)$. The subcollection of these, each one of which contains among its chief points a point of X_a, is an arc‖ C_a, not including its ends. Let the ends of C_a be U and V, let R be any element of C with a chief point in X_r, and let C_u and C_v be the

* *On certain types of plane continua*, p. 809, loc. cit.

† C. Zarankiewicz, *Ueber die Zerschneidungspunkte der zusammenhängender Mengen*, Fundamenta Mathematicae, vol. 12, p. 121, Hilfsatz.

‡ *On certain types of plane continua*, Corollary 2, loc. cit.

§ Defined by C. Carathéodory in his paper, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

‖ N. E. Rutt, *Prime ends and order*, Part 1, §10. This paper has been accepted for publication by the Annals of Mathematics, but is not yet in print.

subarcs of C complementary to R and C_a including their ends, U being contained by C_u and V by C_v. For each value of j, $(j=1, 2, 3, \cdots)$, let S_j be the set intercepted* by X_a and $[X_i]_j$, let M_j be the subcollection of $U+C_a+V$ each element of which is limit of S_j, but not of $\sum_i X_i^j$, let N_j be the subcollection of $U+C_a+V$ each element of which is limit of $\sum_i X_i^j$, and let V_j be the sum of N_j and the complement in $U+C_a+V$ of M_j. If it be supposed that C_u includes every element of C which is limit of any one of $[X_a]$ contained in a series of $[X_i]_j$,† then $V_j \supset V$. Moreover, as $S_1 \subset S_2 \subset \cdots \subset S_i \subset S_{i+1} \subset \cdots$, then $M_1 \subset M_2 \subset \cdots \subset M_1 \subset M_{i-1} \subset \cdots$, and the elements of $[V_j]$ are arcs with a set H_a in common, such that $V_1 \supset V_2 \supset \cdots \supset V_i \supset V_{i+1} \supset \cdots \supset H_a$. If $[N_j]$ is a collection no infinite subset of which has any common element, then there must be a prime end H in H_a which, considered as a collection of domains, includes no element η which does not contain a prime end belonging to some one of $[N_j]$. If H contains a point h of X_a, then h is a point which, although not limit of any one of the point sets $[S_i]$ through H, is nevertheless limit of the collection $[S_j]$ through H. Let $[\tau_i]$ be a monotonic collection of neighborhoods of h whose only common point is h, so chosen with respect to an arbitrary element η of H that there exists an infinite subset $[T_i]$ of $[S_i]$ having, for each i, $\eta \cdot \tau_i \cdot T_i \neq 0$ and $\eta \cdot \tau_{i+1} \cdot T_i = 0$. Under these circumstances, $[X_a]$ contains a set $[Y_i]$, where $Y_i \cdot \tau_i \neq 0$ and $T_i \supset Y_i$, so that h is a limit point of the point set $\sum Y_i$. But $[Y_i]$ is a series, because for each value of i, $(i=1, 2, 3, \cdots)$, T_{i+1} contains Y_{i+1}, whereas T_i does not; and all elements of T_{i+1} not belonging to T_i are separated from X_a by X_r and any element whatever of T_i. For the same reason, $[Y_i]$ is a series in which Y_i and X_r separate Y_{i+1} and X_a. As the point h of X_a is limit of $\sum Y_i$, this is a contradiction,‡ which proves the theorem in this case.

If H contains no point of X_a, then it is V, and although belonging to C_v, is limit of a series like $[Y_i]$ selected from $[X_a]$, not by means of a collection $[\tau_i]$ of neighborhoods of a point, but by means of a chain of domains defining H; and this is also

* For definitions and properties used here see *Prime ends and order*, Part 3, loc. cit.

† *Prime ends and order*, Part 2, §6, loc. cit.

‡ *On certain types of plane continua*, Corollary 2, loc. cit.

contradictory.* On the other hand, in case infinitely many of $[N_i]$ contain the element H of C_a, then H contains a point h of X_a which is shown, much as above, to be limit of a series $[Y_i]$, where Y_i is an element of T_i but not of T_{i-1}; this constitutes a contradiction, as above.

COROLLARY 1. *If $[[X_i]_a]$ is a collection of series no two of which are concurrent, and for every element $[X_i]_a$ of $[[X_i]_a]$ there is a point a of the element X_a of $[X_a]$ different from X_r within every neighborhood of which there is a point of some element of $[X_i]_a$, then $[[X_i]_a]$ is countable.*

If there are elements of $[[X_i]_a]$ opposite in sense, then $[[X_i]_a]$ consists of two subcollections having the property that no pair of elements belonging to the same one can be either concurrent or opposite in sense. Thus, whether or not there are in $[[X_i]_a]$ two elements opposite in sense, the corollary follows easily from Theorem 1 because when there are two such elements each of the two collections mentioned is countable.

3. *An Application.* We shall now give an application of the foregoing results.

THEOREM 3. *If X_a is any element of $[X_a]$, then $Z-X_a$ is the sum of a countable set of continua, each one of which is of type Z.*

Suppose at the outset that X_a contains a point arcwise accessible from the unbounded complementary domain of Z. If $Z-X_a$ is closed, the theorem is true; whereas, if it is not, then there must be a series of elements $[X_i]_1$ of $[X_a]$ each one arcwise accessible from $c_u(Z)$ among whose limit points is a point of X_a. Let $[X_a]_1$ be the subset of $[X_a]$ consisting of all of its elements which are separated from X_a by some pair of the elements $X_r, X_1^1, X_2^1, \cdots, X_i^1, \cdots$. The set $Z-\Sigma X_a^1$ is clearly a continuum Z_1 containing X_a. If Z_1-X_a is not closed, Z_1 contains a subseries $[X_i]_2$ of elements of $[X_a]$, each accessible from $c_u(Z_1)$ with a limit point in X_a; for, if there were no such series, no point of X_a could be limit of $c_u(Z_1)$, and thus no point of X_a could be arcwise accessible from $c_u(Z)$. There is thus a set $[X_a]_2$ and a set $Z_2=Z_1-\Sigma X_a^2$ which is closed and contains X_a. In fact, there are three series $[X_i]_1$, $[X_i]_2$, $[X_i]_3, \cdots$; $[X_a]_1$, $[X_a]_2$,

* *On certain types of plane continua*, Corollary 3, loc. cit.

$[X_\alpha]_3, \cdots$; and $Z_1, Z_2, Z_3, \cdots$. Let $W_\omega = \Pi Z_i$. Clearly W_ω is a continuum of type Z containing $X+X_\alpha$. If $W_\omega - X_\alpha$ is not closed, W_ω contains a series $[X_i]_\omega$ of elements of $[X_\alpha]$, each arcwise accessible from $c_u(W_\omega)$, having a limit point in X_α, so that there is a collection $[X_i]_\omega$ and a continuum Z_ω resembling Z_1. In short, the process described may be continued until a set Z_σ is obtained having $Z_\sigma - X_\alpha$ closed. If Z_σ is not a set that has been obtained in the way that W_ω was obtained, then the order type of $Z_1, Z_2, \cdots, Z_\omega, \cdots, Z_\sigma$ is the same as that of $[X_i]_1, [X_i]_2, \cdots, [X_i]_\omega, \cdots, [X_i]_\sigma$; whereas, if not, then the order type of the collection $[Z_\lambda]$ may be obtained from that of $[[X_i]_\lambda]$ by the addition of the single transfinite ordinal σ. However, in either case, $[Z_\lambda]$ is a countable collection because, owing to Corollary 1, $[[X_i]_\alpha]$ is countable.

The set $Z - X_\alpha$ may now be expressed as the sum of a countable collection of sets as follows. Let the set K_0 be $X+X_\tau+X_1^1$ plus all the elements of $[X_\alpha]$ which are separated from X_α by X_1^1 and X_τ. Let K_1 be $X+X_1^1+X_2^1$ plus all elements of $[X_\alpha]$ separated from both X_α and X_τ by X_1^1 and X_2^1. Let K_n, $(n=2, 3, 4, \cdots)$, be $X+X_n^1+X_{n+1}^1$ plus all elements of $[X_\alpha]$ separated from X_α by X_n^1 and X_{n+1}^1. In general, as to K_λ, if λ is a transfinite ordinal of the form $\mu+n$, where n is a finite positive integer and μ is a transfinite ordinal not of the same form as λ, then let K_λ be $X+X_n^\mu+X_{n+1}^\mu$ plus all elements of $[X_\alpha]$ separated from X_α by X_n^μ and X_{n+1}^μ; while, if λ is not of this form, then let K_λ consist of X and all of $[X_\tau]$, where X_t, any element of $[X_\tau]$, is X_1^λ or any element of $[X_\alpha]$ separated from X_α by X_1^λ, and $[X_i]_\beta$, where $[X_i]_\beta$ is any element of $[[X_i]_\lambda]$ not opposite in sense to $[X_i]_\lambda$ with β a transfinite ordinal preceding λ. The collection $[K_\lambda]$ is clearly countable, since it has the same cardinal number as the collection $[X_0^\lambda]$ of all the elements of $[X_\alpha]$ included in one of $[[X_i]_\lambda]$.

Consider the set K_λ. It is obviously connected. If it is not closed, let l be a limit point of it. Now if $\lambda = \mu+n$, n and μ being as in the paragraph above except that μ may possibly be zero, then $K_\lambda \subset Z_\mu$, so that the point l must belong to some element X_l of $[X_\alpha]$ contained in Z_μ. All but two of the elements of $[X_\alpha]$ in Z_μ belonging to K_λ are separated from X_α by X_{n+1} and X_n, both of these being elements of $[X_\alpha]$ in Z_μ arcwise accessible from $c_u(Z_\mu)$, so, as X_l contains l, it can not be separated from them

by X_n and X_{n+1}. Thus l may exist only if $n=0$. But in this case X_l would have to be separated from any series of $[X_\alpha]$ in K_λ of which it contains a limit by X_1^λ and X_r, both of these being elements of $[X_\alpha]$ in K_λ arcwise accessible from $c_u(Z_\lambda)$. So l can not exist, and K_λ is closed. The statements above apply directly to all except K_0, which is easily seen to be a continuum by similar means. Accordingly, when X_a contains a point arcwise accessible from $c_u(Z)$, the fact that $Z-X_a=\Sigma K_\lambda+(Z_\sigma-X_a)$ verifies the theorem.

In case X_a contains no point arcwise accessible from $c(Z)$, let $[X_i]_1$ be a series of $[X_\alpha]$, such that, for $i=1, 2, 3, \cdots$, X_r and X_{i+1}^1 separate X_a and X_i^1, each element of $[X_i]_1$ is arcwise accessible from $c(Z)$, and there is none of $[X_\alpha]$ arcwise accessible from $c_u(Z)$ together with X_r separating X_a from more than a finite number of $[X_i]_1$. Suppose that, in addition to satisfying requirements specified earlier, X_r has also been selected so as to be separated from X_a by the elements X_b and X_c likewise arcwise accessible from $c_u(Z)$. If we omit all those of $[X_\alpha]$ separated from both X_a and X_r by either X_b or X_c and some element of $[X_i]_1$, a subcontinuum $\overline{Z}_1$ containing X_a results. If $\overline{Z}_1-X_a$ is not closed and X_a contains no point arcwise accessible from $c_u(\overline{Z}_1)$, the step above may be repeated, and, under similar circumstances, may be repeated indefinitely, with an occasional inserted step of finding $\Pi\overline{Z}_i$, until eventually there results a continuum $\overline{Z}_\sigma$ in which either $\overline{Z}_\sigma-X_a$ is closed or X_a contains a point arcwise accessible from $c_u(\overline{Z}_\sigma)$. The collection $[[X_i]_\lambda]$ of series used in determining $\overline{Z}_\sigma$ consists of two subaggregates in each of which no two elements can be concurrent or opposite in sense; hence, from Theorem 1, it follows easily that $[[X_i]_\lambda]$ is countable. Consequently, very much as above, in the case already discussed, it may be seen that $(Z-\overline{Z}_\sigma)+X$ is the sum of a countable set of continua of type Z, so that since $\overline{Z}_\sigma-X_a$ has already been seen to be the sum of a countable set of continua of type Z, then $Z-X_a$ is also.

COROLLARY 2. *If* $[X_n]$ *is a finite subset of* $[X_\alpha]$, *then* $Z-\Sigma X_n$ *is the sum of a countable set of continua each of type* Z.

For $Z-X_1$ is the sum of a countable collection $[K_\alpha]_1$ of the sort required, so consider the distribution of the remaining members of $[X_n]$ among those of $[K_\lambda]_1$. If no more than one of $[X_n]$

is contained in any one of $[K_\lambda]_1$, the corollary follows easily from Theorem 3. If, on the other hand, K_p^1 of $[K_\lambda]_1$ were to contain more than one of $[X_n]$, X_q being that one of $[X_n]$ lowest in subscript which it contains, then $K_p^1 - X_q$ would be the sum of a countable set of continua of type Z, so that, after taking due account of the fact that X_q might belong to two different elements of $[K_\lambda]_1$ (but not to more than two), it would appear that $Z-(X_1+X_q)$ was also sum of a countable set $[K_\lambda]_2$ of continua of type Z. The process can be carried through a finite number of steps to prove the corollary.

COROLLARY 3. *If $[X_\lambda]$ is a countable subset of $[X_\alpha]$ not including X_r, and $[X_\alpha]$ contains a countable collection of pairs of elements, such that each element of every pair contains a point arcwise accessible from $c_u(Z)$, no pair separates from X_r either any element contained in any other pair or more than a finite number of $[X_\lambda]$, and no element of $[X_\lambda]$ is not separated by some pair from X_r, then $Z-\Sigma X_\lambda$ is the sum of a countable collection of continua each of type Z.*

This corollary follows directly from Corollary 2 if we express Z as the sum of a countable collection of continua each of which, except the one containing X_r, consists of X, a pair, and all of $[X_\alpha]$ separated from X_r by the pair.

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HAUSDORFF TRANSFORMATIONS FOR DOUBLE SEQUENCES

BY C. R. ADAMS

1. *Introduction.* The purpose of this note is to extend to double sequences some of the results of Hausdorff's notable papers on methods of summability and moment-sequences.†

Let $\lambda = \| \lambda_{pqmn} \|$, $(p, q, m, n = 0, 1, 2, \cdots)$, be a four-dimensional matrix of real or complex numbers. Then the system of equations

$$A_{pq} = \sum_{m,n=0}^{\infty} \lambda_{pqmn} a_{mn}, \tag{1}$$

if the series all converge, transforms a double sequence $\{a_{mn}\}$ into a new double sequence $\{A_{pq}\}$. Necessary and sufficient conditions‡ that this transformation be convergence-preserving for bounded sequences are the following:

$$\sum_{m,n=0}^{\infty} | \lambda_{pqmn} | \leqq M, \qquad (p, q = 0, 1, 2, \cdots), \tag{A}$$

$$\lim_{p,q \to \infty} \sum_{m,n=0}^{\infty} \lambda_{pqmn} = l, \tag{B}$$

$$\lim_{p,q \to \infty} \lambda_{pqmn} = l_{mn}, \qquad (m, n = 0, 1, 2, \cdots), \tag{C}$$

$$\lim_{p,q \to \infty} \sum_{m=0}^{\infty} | \lambda_{pqmn} - l_{mn} | = 0, \qquad (n = 0, 1, 2, \cdots), \tag{D}$$

† Hausdorff, *Summationsmethoden und Momentfolgen*, I, II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299.

‡ See Robison, *Divergent double sequences and series*, Transactions of this Society, vol. 28 (1926), pp. 50–73, especially pp. 71–72. Such transformations are sometimes, if not always, convergence-preserving for certain unbounded sequences; see Adams, *Transformations of double sequences, with application to Cesàro summability of double series*, this Bulletin, vol. 37 (1931), pp. 741–748; Lösch, *Über den Permanenzsatz gewisser Limitierungsverfahren für Doppelfolgen*, Mathematische Zeitschrift, vol. 34 (1931), pp. 281–290; Adams, *On summability of double series*, Transactions of this Society, vol. 34 (1932), pp. 215–230, hereafter referred to as A; and Agnew, *On summability of double sequences*, American Journal of Mathematics, vol. 54 (1932), pp. 648–656.

$$\text{(E)} \qquad \lim_{p,q\to\infty} \sum_{n=0}^{\infty} |\lambda_{pqmn} - l_{mn}| = 0, \quad (m = 0, 1, 2, \cdots).$$

A matrix λ satisfying these conditions we call, with Hausdorff, a *C-matrix*. Such a matrix defines a transformation (1) which carries a bounded sequence $\{a_{mn}\}$ convergent to α into a bounded sequence $\{A_{pq}\}$ convergent to

$$l\alpha + \sum_{m,n=0}^{\infty} l_{mn}(a_{mn} - \alpha),$$

this double series being always absolutely convergent. For a C-matrix to define a transformation regular for bounded *null* sequences it is necessary and sufficient that all the l_{mn} vanish; in such a case the C-matrix will be called *pure*. For a pure C-matrix to define a transformation regular for *all* bounded sequences it is necessary and sufficient that $l=1$; in this event the C-matrix will be called *normalized*.

Now let ρ be a *fixed* matrix having an inverse and $\mu = \|\mu_{mnkl}\|$ an *arbitrary "diagonal"* matrix; that is,

$$\mu_{mnkl} = 0,$$

for $k \neq m$ or $l \neq n$ or both, so that the only elements $\neq 0$ are $\mu_{mnmn} (m, n = 0, 1, 2, \cdots)$. For simplicity we write

$$\mu_{mnmn} = \mu_{mn}.$$

Henceforth we restrict ourselves to matrices λ of the form†

$$\lambda = \rho^{-1} \cdot \mu \cdot \rho, \tag{2}$$

which by means of the fixed ρ are transformable into diagonal matrices. It is seen at once that *any two such matrices are permutable*; moreover, the proof is immediate that if $\lambda^* = \rho^{-1} \cdot \mu^* \cdot \rho$ is one of the matrices (2) for which the corresponding diagonal matrix μ^* has elements μ^*_{mn} no two of which are equal, then *all* matrices λ permutable with λ^* are of the form (2).

The system of equations (1) may be written in matrix form as

$$A = \lambda \cdot a, \tag{3}$$

where $a = \|a_{mnkl}\|$ and $A = \|A_{pqmn}\|$, with

† By a product such as $\lambda \cdot \mu$ we mean the matrix whose element with indices p, q, k, l is $\sum_{m,n=0}^{\infty} \lambda_{pqmn}\mu_{mnkl}$.

$$a_{mnkl} = \begin{cases} a_{mn} \text{ for } k = l = 0, \\ 0 \text{ otherwise,} \end{cases} \qquad A_{pqmn} = \begin{cases} A_{pq} \text{ for } m = n = 0, \\ 0 \text{ otherwise.} \end{cases}$$

If we set $b = \rho \cdot a$, $B = \rho \cdot A$, the matrix equation (3) becomes

$$B = \mu \cdot b, \qquad \text{or} \qquad B_{mn} = \mu_{mn} b_{mn}, \tag{4}$$

a *multiplication*. To the matrix λ there corresponds a unique diagonal matrix μ, or factor sequence $\{\mu_{mn}\}$, and conversely. If λ is a C-matrix, we call $\{\mu_{mn}\}$ a *C-sequence*; a C-sequence will be said to be pure or normalized according as λ is pure or normalized.

2. *Difference Sequences.* From a double sequence $\{a_{mn}\}$ one may form the quadruple sequence of double differences of various orders,

$$\Delta_{ij} a_{mn} = \sum_{k,l=0}^{i,j} (-1)^{k+l} \binom{i}{k} \binom{j}{l} a_{m+k,n+l}. \tag{5}$$

The recursion formulas

$$\begin{aligned} \Delta_{ij} a_{mn} &= \Delta_{ij} a_{m+1,n} + \Delta_{i+1,j} a_{mn} \\ &= \Delta_{ij} a_{m,n+1} + \Delta_{i,j+1} a_{mn} \end{aligned} \tag{6}$$

may at once be derived. For brevity we call the double sequence of numbers

$$\Delta_{ij} a_{00} = b_{ij} = \Delta_{00} b_{ij}$$

the difference sequence of the a_{mn}, and from (6) readily follows the relation

$$\Delta_{ij} a_{mn} = \Delta_{mn} b_{ij},$$

so that the $a_{mn} = \Delta_{00} a_{mn}$ also constitute the difference sequence of the b_{ij}. The matrix associated with (5) is the matrix of the Euler transformation for double sequences,

$$E = E' \odot E',$$

where E' denotes the Euler matrix for simple sequence transformations and a notation employed in A is used.† By Theorem

† That is, if each element a_{mnkl} of a four-dimensional matrix a is equal to $a'_{mk} \cdot a''_{nl}$, where these factors are the elements of two-dimensional matrices a' and a'', we write $a = a' \odot a''$.

7 of A, E is its own inverse. We now choose E to be the fixed matrix ρ introduced above, and hereafter consider only such systems of equations (1) as have their difference sequences b_{mn}, B_{mn} in the multiplicative form (4).

We may express the equations (4) in terms of a_{mn}, A_{pq} and obtain without difficulty from $A = E \cdot B$ the relation

$$A_{pq} = \sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j}[\Delta_{p-i,q-j}\mu_{ij}]a_{ij}.$$

Setting all the $a_{ij} = 1$, we have

$$(7) \qquad \sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} = \Delta_{00}\mu_{00} = \mu_{00}.$$

Then the conditions (A)-(E) become

$$(\alpha) \qquad \sum_{i,j=0}^{p,q} \binom{p}{i}\binom{q}{j}|\Delta_{p-i,q-j}\mu_{ij}| = M_{pq} \leqq M, \qquad (p, q = 0, 1, 2, \cdots),$$

$$(\gamma) \qquad \lim_{p,q\to\infty} \binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} = l_{ij}, \qquad (i, j = 0, 1, 2, \cdots),$$

$$(\delta) \qquad \lim_{p,q\to\infty} \sum_{i=0}^{p} \left|\binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} - l_{ij}\right| = 0, \qquad (j = 0, 1, 2, \cdots),$$

$$(\epsilon) \qquad \lim_{p,q\to\infty} \sum_{j=0}^{q} \left|\binom{p}{i}\binom{q}{j}\Delta_{p-i,q-j}\mu_{ij} - l_{ij}\right| = 0, \qquad (i = 0, 1, 2, \cdots),$$

the condition (β) corresponding to (B) being automatically satisfied because of (7). These conditions (α)-(ϵ) are thus necessary and sufficient for a C-sequence. A C-sequence is pure if and only if all the l_{ij} vanish; a pure C-sequence is normalized if and only if $\mu_{00} = 1$.

The corresponding conditions, necessary and sufficient for a *simple* sequence $\{\mu_m\}$ to be a C-sequence, are

$$(\mathfrak{A}) \qquad \sum_{i=0}^{p} \binom{p}{i} |\Delta_{p-i}\mu_i| = M_p \leqq M, \qquad (M_p \to M),$$

$$(\mathfrak{C}) \qquad \lim_{p\to\infty} \binom{p}{i} \Delta_{p-i}\mu_i = l_i, \qquad (i = 0, 1, 2, \cdots).$$

Of these the first implies the second with $l_i=0$ for $i>0$. It is therefore natural to inquire to what extent condition (α) implies the three remaining conditions (γ)-(ϵ). The complete answer to this question will be given in §4, although even in §3 it will become apparent that (α) does not imply (δ) and (ϵ).

3. *C-Sequences of Product Type.* An interesting class of matrices λ of the form (2) is that for which the diagonal matrix corresponding to λ is the *product* (see the last footnote above) of (simple) diagonal matrices. The importance of this class is sufficiently indicated by the fact that, by Theorem 6 of A, this class is identical with the class of matrices $\lambda=\lambda' \odot \lambda''$, in which λ' and λ'' are transformable by E' into diagonal matrices.

If $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, we clearly have

$$\Delta_{p-i,q-j}\mu_{ij} = \Delta_{p-i}\mu_i' \Delta_{q-j}\mu_j'' .$$

The following theorems may now be established easily.

THEOREM 1. *A double sequence* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, *where* $\{\mu_m'\}$ *and* $\{\mu_n''\}$ *are (simple) C-sequences, is a (double) C-sequence if and only if the following condition is satisfied*:

$$(8) \qquad |l_0'|\,(M'' - |l_0''|) = |l_0''|\,(M' - |l_0'|) = 0.$$

Here the M and l_0 of $(\mathfrak{A})$ and $(\mathfrak{C})$ are primed to agree with the priming of the μ's. Any double sequence thus factorable into simple C-sequences obviously satisfies (α), but it will satisfy both (δ) and (ϵ) if and only if (8) is also fulfilled.

THEOREM 2. *A double sequence* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$, *where* $\{\mu_m'\}$ *and* $\{\mu_n''\}$ *are (simple) C-sequences, is a pure (double) C-sequence if and only if one of its factors is a pure C-sequence and the other is a pure C-sequence or satisfies the condition* $M = |l_0|$.

THEOREM 3. *If* $\{\mu_{mn}\} = \{\mu_m' \mu_n''\}$ *is a (double) C-sequence that is not pure, both of its factors are C-sequences, neither factor is pure, and* $l_0' l_0''$ *equals* l_{00}.

THEOREM 4. *If* $\{\mu_{mn}\} = \{\mu_m'\mu_n''\}$ *is a pure (double) C-sequence, not all of whose elements are zero, both of its factors are C-sequences and at least one of them is pure.*

4. *The Relation Between C-Sequences and Moment-Sequences.* Let $\chi(u, v)$ be a function which in the square $U(0 \leqq u \leqq 1, 0 \leqq v \leqq 1)$ is of bounded variation in the sense of Hardy-Krause; then the sequence $\{\mu_{mn}\}$, where

$$(9) \qquad \mu_{mn} = \int_0^1 \int_0^1 u^m v^n d_u d_v \chi(u, v), \qquad (m, n = 0, 1, 2, \cdots),$$

may be termed a (double) *moment-sequence.*† According to a theorem of Hildebrandt and Schoenberg,‡ if $\{\mu_{mn}\}$ is any double sequence satisfying condition (α) for $q = p$, there exists a function $\chi(u, v)$ which generates this sequence, and conversely; hence we have the following theorem.

THEOREM 5. *Every C-sequence is a moment-sequence.*

Clearly not every moment-sequence is a C-sequence; for otherwise, by Hildebrandt and Schoenberg's theorem, condition (α) for $q = p$ would imply both (δ) and (ϵ), and we have already seen that this is not so. We therefore seek to determine under what circumstances a moment-sequence is a C-sequence.

It should be observed first that since $\chi(u, v)$ can always be expressed as the difference between two functions $\chi_1(u, v)$ and $\chi_2(u, v)$, each of which is bounded, is non-decreasing in x alone and in y alone, and satisfies the condition $\chi_i(u_1, v_1) - \chi_i(u_1, v_2) - \chi_i(u_2, v_1) + \chi_i(u_2, v_2) \geqq 0$ for $u_2 > u_1, v_2 > v_1$; and since the difference (or sum) of two C-sequences is a C-sequence, no loss of generality results from assuming $\chi(u, v)$ to be of this monotonic character. Then we have, for all i, j, m, n,

† Although apparently it is more general to assume only that $\chi(u, v)$ is of bounded variation in the sense of Vitali, this is not actually so; for if $\chi(u, v)$ is of bounded variation in the Vitali sense, there always exists a related function $\chi'(u, v)$, where $\chi'(u, v) = \chi(u, v) - \chi(u, 0) - \chi(0, v) + \chi(0, 0)$, such that the Riemann-Stieltjes integral (9) taken with respect to $\chi'(u, v)$ has the same value, and $\chi'(u, v)$ is of bounded variation in the sense of Hardy-Krause. Moreover, $\chi'(u, v)$ vanishes along $u = 0$ and $v = 0$; hence it would be no real restriction to assume $\chi(u, 0) \equiv \chi(0, v) \equiv 0$. We do not make this assumption, however, for reasons of symmetry.

‡ Hildebrandt and Schoenberg, *On linear functional operations and the moment problem for a finite interval in one or several dimensions*, Annals of Mathematics, (2), vol. 34 (1933), pp. 317–328, Theorem 1.

$$\Delta_{ij}\mu_{mn} = \int_0^1 \int_0^1 u^m v^n (1-u)^i (1-v)^j d_u d_v \chi(u, v) \geqq 0,$$

so that $\{\mu_{mn}\}$ is *completely monotonic*, and by (7), condition (α) is satisfied for all p, q. We shall now show that the remaining conditions (γ)-(ϵ) are fulfilled *in case* $\chi(u, v)$ *satisfies the continuity condition*

$$\begin{aligned} &\chi(u, +0) = \chi(u, 0), \quad \chi(u, +0) = \lim_{v \to 0} \chi(u, v), \quad 0 \leqq u \leqq 1; \\ &\chi(+0, v) = \chi(0, v), \quad \chi(+0, v) = \lim_{u \to 0} \chi(u, v), \quad 0 \leqq v \leqq 1. \end{aligned} \tag{10}$$

For all values of u and v in U we have

$$\begin{aligned} \binom{p}{i} u^i (1-u)^{p-i} &\leqq \sum_{i=0}^{p} \binom{p}{i} u^i (1-u)^{p-i} \\ &= [u + (1-u)]^p = 1, \end{aligned} \tag{11}$$

together with a like relation in v. Hence, for $0 < \delta < 1$, we obtain

$$\begin{aligned} \binom{p}{i}\binom{q}{j} \Delta_{p-i, q-j}\mu_{ij} \leqq & \int_0^\delta \int_0^1 d_u d_v \chi(u, v) \\ &+ \binom{p}{i} (1-\delta)^{p-i} \int_\delta^1 \int_0^1 d_u d_v \chi(u, v). \end{aligned}$$

Consequently, for each δ, we have

$$\begin{aligned} \overline{\lim_{p, q \to \infty}} \binom{p}{i}\binom{q}{j} \Delta_{p-i, q-j}\mu_{ij} & \\ \leqq \chi(\delta, 1) - \chi(0, 1) - \chi(\delta, 0) + \chi(0, 0), & \end{aligned}$$

and so, by virtue of (10), the limit exists and equals zero for each i and each j. Thus condition (γ) is satisfied with all $l_{ij} = 0$. That (δ) and (ϵ) are fulfilled then follows in a precisely similar manner, proper account being taken of the relations (11).

When $\chi(u, v)$ does not satisfy condition (10), let

$$\begin{aligned} J_1(u) &= \chi(u, +0) - \chi(u, 0), \quad u > 0; \\ J_2(v) &= \chi(+0, v) - \chi(0, v), \quad v > 0; \\ J_1(0) = J_2(0) &= \lim_{\substack{0 \neq u \to 0 \\ 0 \neq v \to 0}} \chi(u, v) - \chi(0\ 0). \end{aligned} \tag{12}$$

Since $\chi(u, v)$ is bounded and monotonic, all the limits involved exist, and $J_1(u)$ and $J_2(v)$ are functions of bounded variation. Moreover, condition (10) is satisfied by the function

$$\chi^*(u, v) = \chi(u, v) + J(u, v),$$

where

$$J(u, v) = \begin{cases} J_1(u) & \text{for } v = 0, \\ J_2(v) & \text{for } u = 0, \\ 0 & \text{for } u, v > 0. \end{cases}$$

Therefore it remains only for us to investigate the sequence $\{\mu_{mn}\}$ generated by $J(u, v)$.

From the definition of the integral in (9), we have at once

$$\begin{aligned} \mu_{00} &= J_1(0) - J_1(1) - J_2(1), \\ \mu_{m0} &= -\int_0^1 u^m dJ_1(u), \qquad (m = 1, 2, 3, \cdots), \\ \mu_{0n} &= -\int_0^1 v^n dJ_2(v), \qquad (n = 1, 2, 3, \cdots), \\ \mu_{mn} &= 0, \qquad (m, n = 1, 2, 3, \cdots). \end{aligned} \tag{13}$$

Thus $\{\mu_{m0}\}$, $(m=0, 1, 2, \cdots)$, and $\{\mu_{0n}\}$, $(n=0, 1, 2, \cdots)$, are both (simple) C-sequences. By (13) we have

$$M_{pq} = \sum_{i=0}^{p} \binom{p}{i} |\Delta_{p-i,0}\mu_{i0}| + \sum_{j=0}^{q} \binom{q}{j} |\Delta_{0,q-j}\mu_{0j}| - |\Delta_{pq}\mu_{00}|.$$

Both sums are bounded, and the remaining term on the right is also bounded, since

$$\Delta_{pq}\mu_{00} = \Delta_{p0}\mu_{00} + \Delta_{0q}\mu_{00} - \mu_{00} \tag{14}$$

is the sum of a term depending upon p alone and a term depending upon q alone, each of which tends to a limit as p and q become infinite. Therefore (α) is satisfied.

Condition (γ) is obviously satisfied for $i, j>0$, with $l_{ij}=0$. For $i=0, j>0$, the quantity in question reduces to

$$\binom{q}{j}\Delta_{0,q-j}\mu_{0j}, \tag{15}$$

which tends to zero with $1/q$; for $i>0$, $j=0$, the situation is similar. For $i=j=0$, the quantity in question reduces to (14), which, as we have already observed, tends to a limit. Therefore (γ) is fulfilled, with $l_{ij}=0$ for $i, j\neq 0, 0$ and

$$l_{00} = \lim_{p\to\infty} \Delta_{p0}\mu_{00} + \lim_{q\to\infty} \Delta_{0q}\mu_{00} - \mu_{00}.$$

For $j>0$, the sum in (δ) reduces to the absolute value of (15), which tends to zero with $1/q$. For $j=0$, the sum reduces to

$$|\Delta_{pq}\mu_{00} - l_{00}| + \sum_{i=0}^{p}\binom{p}{i}|\Delta_{p-i,0}\mu_{i0}| - |\Delta_{p0}\mu_{00}|,$$

of which the first term tends to zero but the remaining part in general does not.

For $i>0$, the sum in (ϵ) reduces to

$$\binom{p}{i}|\Delta_{p-i,0}\mu_{i0}|,$$

which tends to zero with $1/p$. For $i=0$, the sum reduces to

$$|\Delta_{pq}\mu_{00} - l_{00}| + \sum_{j=0}^{q}\binom{q}{j}|\Delta_{0,q-j}\mu_{0j}| - |\Delta_{0q}\mu_{00}|,$$

of which the first term tends to zero but again the remaining part in general does not. The results of this discussion may be summed up as follows.

THEOREM 6. *Let $J_1(u)$ and $J_2(v)$ represent the jump of $\chi(u, v)$ at the sides $v=0$ and $u=0$, respectively, of the square U (for orthogonal approach except at (0, 0), where the approach is made in an arbitrary manner from the interior of U). $J_1(u)$ and $J_2(v)$ are functions of bounded variation and so generate (simple) moment-sequences which we may denote by $\{j_m^{(1)}\}$ and $\{j_m^{(2)}\}$, respectively. The sequences $j_0^{(1)}-J_2(1)$, $j_1^{(1)}$, $j_2^{(1)}$, $\cdots$ and $j_0^{(2)}-J_1(1)=j_0^{(1)}-J_2(1)$, $j_1^{(2)}$, $j_2^{(2)}$, $\cdots$, are also moment-sequences and therefore C-sequences; let the M of $(\mathfrak{A})$ and the l_0 of $(\mathfrak{C})$ associated with these sequences be denoted respectively by $M^{(1)}$, $l_0^{(1)}$ and $M^{(2)}$, $l_0^{(2)}$. Then a (double) moment-sequence is a (double) C-sequence if and only if we have*

Since $\chi(u, v)$ is bounded and monotonic, all the limits involved exist, and $J_1(u)$ and $J_2(v)$ are functions of bounded variation. Moreover, condition (10) is satisfied by the function

$$\chi^*(u, v) = \chi(u, v) + J(u, v),$$

where

$$J(u, v) = \begin{cases} J_1(u) & \text{for } v = 0, \\ J_2(v) & \text{for } u = 0, \\ 0 & \text{for } u, v > 0. \end{cases}$$

Therefore it remains only for us to investigate the sequence $\{\mu_{mn}\}$ generated by $J(u, v)$.

From the definition of the integral in (9), we have at once

$$\begin{aligned} \mu_{00} &= J_1(0) - J_1(1) - J_2(1), \\ \mu_{m0} &= -\int_0^1 u^m dJ_1(u), \qquad (m = 1, 2, 3, \cdots), \\ \mu_{0n} &= -\int_0^1 v^n dJ_2(v), \qquad (n = 1, 2, 3, \cdots), \\ \mu_{mn} &= 0, \qquad (m, n = 1, 2, 3, \cdots). \end{aligned} \tag{13}$$

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$$M_{pq} = \sum_{i=0}^{p} \binom{p}{i} |\Delta_{p-i,0}\mu_{i0}| + \sum_{j=0}^{q} \binom{q}{j} |\Delta_{0,q-j}\mu_{0j}| - |\Delta_{pq}\mu_{00}|.$$

Both sums are bounded, and the remaining term on the right is also bounded, since

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Condition (γ) is obviously satisfied for $i, j>0$, with $l_{ij}=0$. For $i=0, j>0$, the quantity in question reduces to

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which tends to zero with $1/q$; for $i>0$, $j=0$, the situation is similar. For $i=j=0$, the quantity in question reduces to (14), which, as we have already observed, tends to a limit. Therefore (γ) is fulfilled, with $l_{ij}=0$ for $i, j\neq 0, 0$ and

$$l_{00} = \lim_{p\to\infty} \Delta_{p0}\mu_{00} + \lim_{q\to\infty} \Delta_{0q}\mu_{00} - \mu_{00}.$$

For $j>0$, the sum in (δ) reduces to the absolute value of (15), which tends to zero with $1/q$. For $j=0$, the sum reduces to

$$|\Delta_{pq}\mu_{00} - l_{00}| + \sum_{i=0}^{p}\binom{p}{i}|\Delta_{p-i,0}\mu_{i0}| - |\Delta_{p0}\mu_{00}|,$$

of which the first term tends to zero but the remaining part in general does not.

For $i>0$, the sum in (ϵ) reduces to

$$\binom{p}{i}|\Delta_{p-i,0}\mu_{i0}|,$$

which tends to zero with $1/p$. For $i=0$, the sum reduces to

$$|\Delta_{pq}\mu_{00} - l_{00}| + \sum_{j=0}^{q}\binom{q}{j}|\Delta_{0,q-j}\mu_{0j}| - |\Delta_{0q}\mu_{00}|,$$

of which the first term tends to zero but again the remaining part in general does not. The results of this discussion may be summed up as follows.

THEOREM 6. *Let $J_1(u)$ and $J_2(v)$ represent the jump of $\chi(u, v)$ at the sides $v=0$ and $u=0$, respectively, of the square U (for orthogonal approach except at $(0, 0)$, where the approach is made in an arbitrary manner from the interior of U). $J_1(u)$ and $J_2(v)$ are functions of bounded variation and so generate (simple) moment-sequences which we may denote by $\{j_m^{(1)}\}$ and $\{j_m^{(2)}\}$, respectively. The sequences $j_0^{(1)}-J_2(1)$, $j_1^{(1)}$, $j_2^{(1)}$, $\cdots$ and $j_0^{(2)}-J_1(1)=j_0^{(1)}-J_2(1)$, $j_1^{(2)}$, $j_2^{(2)}$, $\cdots$, are also moment-sequences and therefore C-sequences; let the M of $(\mathfrak{A})$ and the l_0 of $(\mathfrak{E})$ associated with these sequences be denoted respectively by $M^{(1)}$, $l_0^{(1)}$ and $M^{(2)}$, $l_0^{(2)}$. Then a (double) moment-sequence is a (double) C-sequence if and only if we have*

$$M^{(1)} - |l_0^{(1)}| = M^{(2)} - |l_0^{(2)}| = 0. \tag{16}$$

A (double) moment-sequence is a pure (double) C-sequence if and only if in addition to (16) *we have*

$$l_0^{(1)} + l_0^{(2)} - j_0^{(1)} + J_2(1) = 0.$$

In particular we note that if $J_1(u) \equiv J_2(v) \equiv 0$, $\chi(u, v)$ generates a pure C-sequence.

THEOREM 7. *Condition* (α) *for* $\varrho = p$ *implies the entire set of conditions* (α)-(ϵ), *including* $l_{ij} = 0$ *for* $i,\ j \neq 0, 0$, *with the exception of* (δ) *for* $j = 0$ *and* (ϵ) *for* $i = 0$.

Although in this section it has been tacitly assumed that the sequences considered are real, the extension of the results to complex sequences is immediate.

BROWN UNIVERSITY

ERRATA

This Bulletin, vol. 38, No. 12 (Dec., 1932):

Page 841, first formula: inside of the large parentheses in the denominator, the numerator of the small fraction should be n instead of 1.

Page 847, equation (11): the quantity c/2 should be added to the left-handed side.

Page 847, last line: the second and third integrals should be preceded by the negative sign.

This Bulletin, vol. 39, No. 1 (Jan., 1933):

Page 18, line 8: *read* Kline *in place of* Kine.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THIRTY-NINTH SUMMER MEETING, CHICAGO, June 19-24, 1933.

This meeting will be in conjunction with the meeting of the American Association for the Advancement of Science at the Century of Progress Exposition, but most of the actual sessions will be at the University of Chicago. The Mathematical Association of America will participate, as usual, and will hold separate meetings on Tuesday and Thursday afternoons. A dinner of mathematicians and their guests is scheduled for Friday evening. Headquarters for mathematicians will be at Judson Court, a dormitory of the University of Chicago, at 60th St. and Ellis Avenue.

Two symposia have been arranged. At one of these, devoted to geometry, Professors Tullio Levi-Civita and Enrico Bompiani of the University of Rome, Professor W. C. Graustein of Harvard University, and others, will speak. At the other symposium, which will be devoted to analysis, Professor Lipot Fejer of the University of Budapest, Professor C. N. Moore of the University of Cincinnati, and others, will speak. Professor Fejer's topic will be *The infinite sequences arising in the theories of harmonic analysis, of interpolation, and of mechanical quadratures.* There will be several additional invited addresses during the week. Professor Levi-Civita will present a paper entitled *On some mathematical aspects of the new mechanics*, and, at the same session, Professor G. D. Birkhoff of Harvard University will speak on *Quantum mechanics and asymptotic series.* This session will probably be a joint session with the American Physical Society. Professor L. E. Dickson of the University of Chicago will give an address entitled *Recent progress in additive number theory.*

Professors Levi-Civita, Bompiani, and Fejer are the guests of the Century of Progress Exposition and the American Association for the Advancement of Science. The meetings at which they speak, and certain others, will be joint sessions of the American Mathematical Society and Section A of the A.A.A.S.

Titles and abstracts of papers to be presented at this meeting should be in the hands of Associate Secretary Mark H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than May 20. Abstracts received before May 6 will appear in the May issue of this BULLETIN.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1932, p. 66), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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